System Control Engineering

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Text: Feedback Systems:
— An Introduction for Scientists and Engineers —
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Text: http://www.cds.caltech.edu/~murray/amwiki/index.php/Version_2.11b
Web: http://www.ic.is.tohoku.ac.jp/en/SystemControlEngineering/

Today's topics

- Global Stability
- Analysis of Linear Time-Invariant Systems
- Lyapunov's indirect method
- Exercises

- Feedback Systems
- Design of Feedback Law
- Backstepping

Global Stability

Asymptotic Stability

- **Local Stability:** The equilibrium x_e is said to be **stable** if

$$\|x(t) - x_e\| < \epsilon, \quad \text{provided that} \quad \|x(0) - x_e\| < \delta$$

  Starting from δ **neighbor** of x_e, the solution will remain ϵ **neighbor** of x_e.

- **Local Asymptotic Stability:** The solution not only stays within ϵ but also converges to x_e in the limit.

Asymptotic Stability in the Large

- When the equilibrium is asymptotically stable, it is often important to know under what conditions an initial state will converge to the equilibrium point.
- In the best possible case, **any initial state** will converge to the equilibrium point.
- An equilibrium point that has this property is said to be **globally asymptotically stable**, or **asymptotically stable in the large**.

Asymptotic Stability in the Large

- **Definition 3.8:** Let $V : D \to \mathbb{R}$ be a continuously differentiable function. Then $V(x)$ is said to be **radially unbounded** if

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty$$

- **Theorem 3.8 Global Asymptotic Stability:** Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, $f : D \to \mathbb{R}^n$, and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

  (i) $V(0) = 0$

  (ii) $V(x) > 0, \quad \forall x \neq 0$

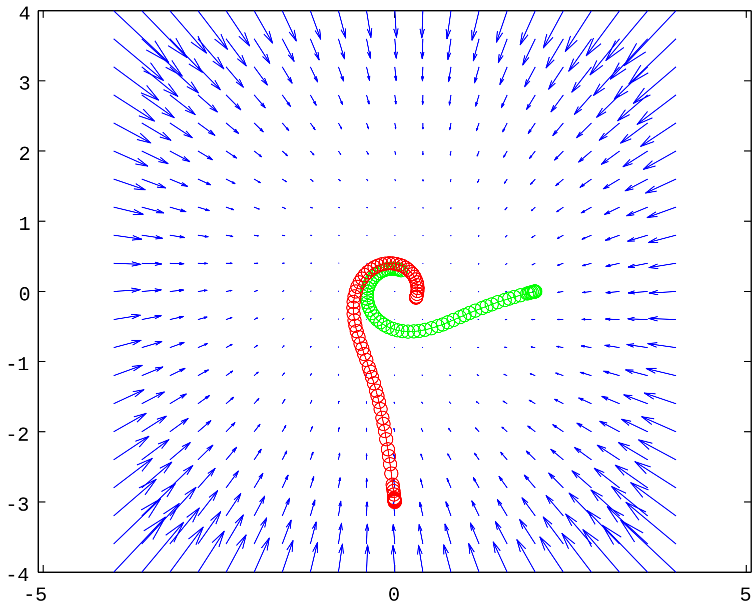
  (iii) $V(x)$ is radially unbounded

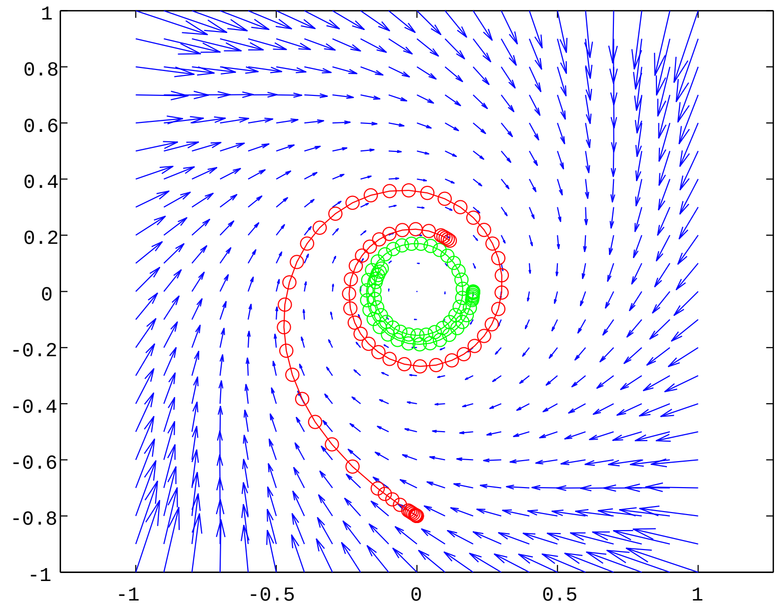
  (iv) $\dot{V} < 0, \quad \forall x \neq 0$

  then $x = 0$ is **globally asymptotically stable**.

Example

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$





$(3, 0), (0, -3)$, $(0.2, 0), (0, -2)$

Example

- Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2)\end{aligned}$$

- To study the equilibrium point at the origin, we define $V(x) = x_1^2 + x_2^2$. Then we have

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x} f(x)\\ &= 2[x_1, x_2][x_2 - x_1(x_1^2 + x_2^2), -x_1 - x_2(x_1^2 + x_2^2)]^T\\ &= -2(x_1^2 + x_2^2)^2.\end{aligned}$$

- Thus, $V(x) > 0$ and $\dot{V} < 0$ for all x. Moreover, since V is radially unbounded, it follows that the origin is globally asymptotically stable.

Example 4.4 Inverted pendulum

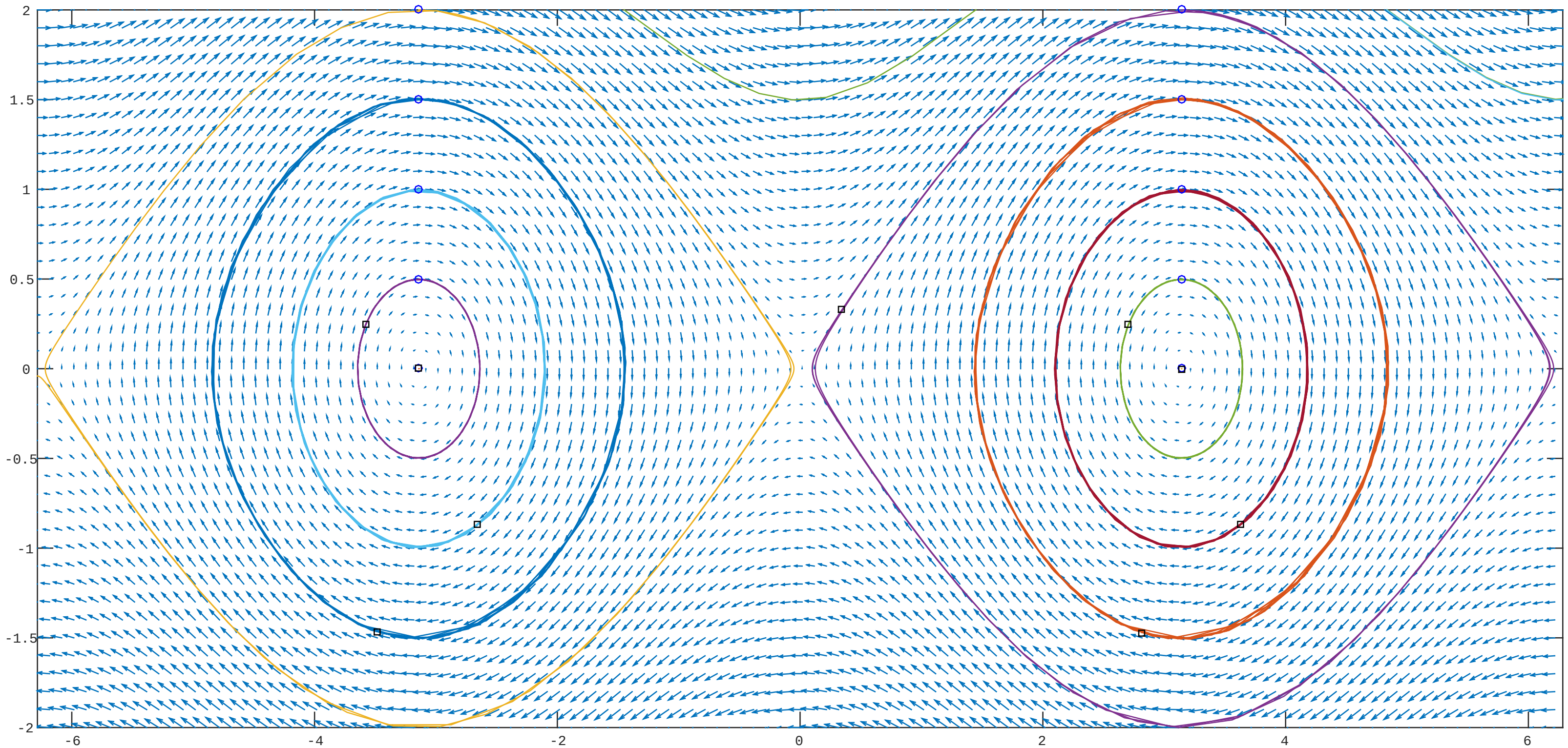
- With assumption that $mg\ell/J_t = 1$ and $m/J_t = 1$, the dynamics (equation (2.10)) become

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - cx_2 + u\cos x_1 \end{bmatrix}, \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

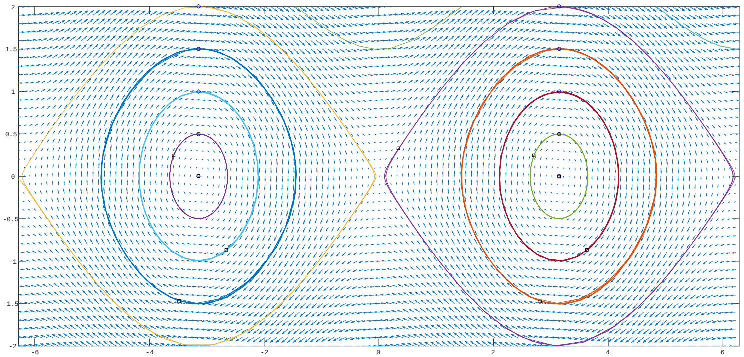
- This is a nonlinear time-independent system of second order.
- The equilibrium points are

$$x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}$$

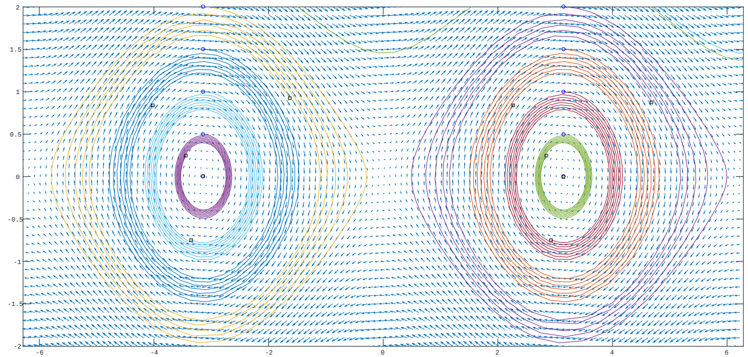
Inverted pendulum phase portrait



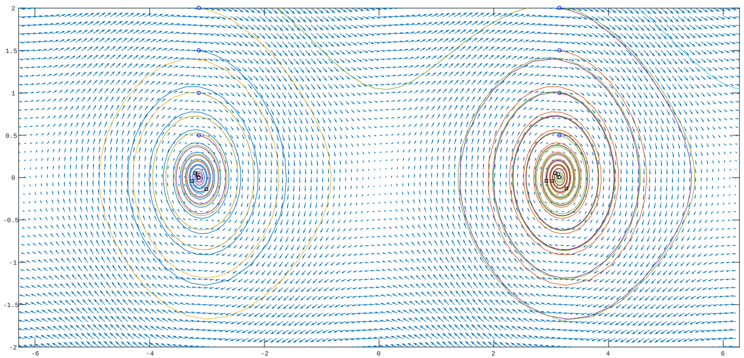
Inverted pendulum phase portrait



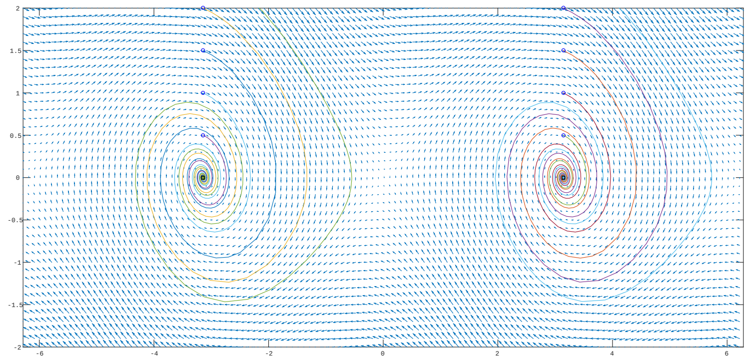
$c = 0,$



$c = 0.01$



$c = 0.1,$



$c = 0.3$

Analysis of Linear Time-Invariant Systems

Stability of LTI System

- LTI system

$$\dot{x} = Ax, \qquad A \in \mathbb{R}^{n \times n}, \qquad x(0) = x_0$$

is **stable** if and only if

all eigenvalues λ of A satisfy Re $\lambda_i \leq 0$ $(i{=}1, \ldots, n)$.

- The equilibrium point $x = 0$ is **exponentially stable** if and only if Re $\lambda_i < 0$ for all i.

- Remember that for nonlinear systems

Exponentially stable $\rightarrow$ Asymptotycally stable $\rightarrow$ Stable

but for linear systems, 0 is the only one equilibrium point and if it is stable then it is exponentially stable.

Stability of LTI System

- Lyapunov function candidate

  $V(x) = x^T P x, \quad P \in \mathbb{R}^{n \times n}$ is positive definite and symmetric

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x$$

- V is Lyapunov function if Q is positive definite

$$Q = -(A^T P + PA)$$

  and this equation is called **Lyapnov equation**.

Stability Check by Lyapunov

(i) Choose an arbitrary symmetric, positive definite matrix Q.

(ii) Find P that satisfies $Q = -(A^T P + PA)$ and verify that P is positive definite.

Lyapunov Theorem for Linear Systems

- **Theorem 3.10:** The eigenvalues λ_i of a matrix A satisfy Re $\lambda_i < 0$ if and only if for any given symmetric positive definite matrix Q there exists a unique positive definite symmetric matrix P satisfying the Lyapunov equation $Q = -(A^T P + PA)$.

Lyapunov's indirect method

Stability analysis via linear approximation

Linearization of Nonlinear Systems

- Consider the nonlinear system

$$\dot{x} = f(x), \qquad f : D \to \mathbb{R}^n$$

and assume that $x = x_e \in D$ is an equilibrium point.

- Taylor series expansion about the equilibrium

$$f(x) = f(x_e) + \left.\frac{\partial f}{\partial x}\right|_{x=x_e} (x - x_e) + \text{h.o.ts}$$

- Neglecting the h.o.ts and recalling $f(x_e) = 0$, we have

$$f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=x_e} (x - x_e)$$

- Now defining

$$\bar{x} = x - x_e, \quad \dot{\bar{x}} = \dot{x}, \quad A = \left.\frac{\partial f}{\partial x}\right|_{x=x_e} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}$$

we have $\dot{\bar{x}} = A\bar{x}$.

Lyapunov's Indirect Method

- **Theorem 3.11** Let $x = 0$ be an equilibrium point for a nonlinear system $\dot{x} = f(x)$. Assume that A is a matrix obtained by liniarization. Then if the eigenvalues λ_i of the matrix A satisfy Re $\lambda_i < 0$, the origin is an exponentially stable equilibrium point.

Exercises

- Consider the following dynamical system:

$$(a) \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + x_1^3 - x_2 \end{cases}$$

$$(b) \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - 2\tan^{-1}(x_1 + x_2) \end{cases}$$

$$(c) \quad \begin{cases} \dot{x}_1 = \frac{2}{3}x_2 \\ \dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2) \end{cases}$$

(a) Find all of its equilibrium points.

(b) Find the linear approximation about each equilibrium point, find the eigenvalues of the resulting A matrix and classify the stability of each equilibrium point.

(c) Construct the phase portrait of each nonlinear system and discuss the qualitative behavior of the system.

(d) Construct the phase portrait of the linearized approximations. Discuss the "accuracy" of the approximations.

Feedback Systems

Feedback Systems

- Consider the system

$$\dot{x} = f(x, u)$$

and assume that the origin $x = 0$ is an equilibrium point of the unforced system $\dot{x} = f(x, 0)$.

- Suppose that input u is obtained using a state feedback

$$u = \phi(x).$$

- Substituting u into $\dot{x}$ yields a unforced system

$$\dot{x} = f(x, \phi(x))$$

Feedback Linearization

- **Example 5.1** Consider the first order system

$$\dot{x} = ax^2 + u$$

Is this system stable?

- We look for a state feedback $u = \phi(x)$ that make the equilibrium point at the origin "asymptotically stabe."
- An obvious way is to **cancels** the nonlinear term

$$u = -ax^2 - x$$

to obtain

$$\dot{x} = -x$$

which is **linear** and grobally asymptotically stable.

Feedback Linearization

- It is based on exact cancellation of the nonlinear term ax^2.
- This is undesirable since in practice system parameters such as a are never known exactly.
- Even if the parameters are not exact, the system can be stabilized.
- But the stability is local because of the presence of the term $(a - \bar{a})x^2$, where a is the true value and $\bar{a}$ is the actual value used in the feedback law.
- Cancelling “all” nonlinear terms may not be a good idea because the nonlinearities are not necessarily bad.

Feedback Linearization

- **Example 5.2** Consider the system given by

$$\dot{x} = ax^2 - x^3 + u$$

and exact cancellation law is

$$u = u_1 = -ax^2 + x^3 - x$$

which leads to

$$\dot{x} = -x.$$

Feedback Linearization

- The presence of terms of the form x^i with i even (偶数のi) on a dynamical equation is never desirable. Indeed, even powers of x do not discriminate sign of the variable x and thus have a **destabilizing** effect that should be avoided whenever possible.
- Terms of the form $-x^j$ with j odd (奇数のj), on the other hand, greatly contribute to the feedback law by providing additional damping for large values of x and are usually **beneficial**.
- At the same time, notice that the cancellation of the term x^3 was achieved by incorporating the term x^3 in the feedback law. The presence of this term in u can lead to very large values of the input. In practice it may cause actuator saturation. The presence of the term x^3 on the input u is **not desirable**.

Design of Feedback Law

Design of Feedback Law

- Given the system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}, \quad f(0,0) = 0$$

we proceed to find a feedback law of the form

$$u = \phi(x)$$

such that the feedback system

$$\dot{x} = f(x, \phi(x))$$

has an asymptotically stable equilibrium at the origin.

Design Policy

- To show that this is the case, we will construct a function $V_1(x) : D \to \mathbb{R}$ satisfying
  (i) $V_1(0) = 0$, and $V_1(x)$ is positive definite in $D - \{0\}$.
  (ii) $\dot{V}_1(x)$ is negative definite along the solutions of $\dot{x} = f(x, \phi(x))$. Moreover, there exist a positive definite function $V_2(x) : D \to \mathbb{R}^+$ such that

$$\dot{V}_1(x) = \frac{\partial V_1}{\partial x} f(x, \phi(x)) \leq -V_2(x), \quad \forall x \in D$$

- Clearly, if $D = \mathbb{R}^n$ and V_1 is radially unbounded, then the origin is globally asymptotically stable.

Example

- Consider again the system

$$\dot{x} = ax^2 - x^3 + u$$

- Define $V_1(x) = \frac{1}{2}x^2$ and compute $\dot{V}$ to obtain

$$\dot{V}_1 = x \cdot f(x,u) = ax^3 - x^4 + xu.$$

- In the previous example we chose $u = u_1 = -ax^2 + x^3 - x$.
- In this case, we have

$$\dot{V}_1 = -x^2 = -V_2(x)$$

and requirement (ii) above is satisfied.

Example

- When we are not happy with the previous example, we modify the function V_2 as follows

$$\dot{V}_1 = ax^3 - x^4 + xu \le -V_2(x) = -(x^4 + x^2)$$

- In this case we must have

$$\begin{aligned} ax^3 - x^4 + xu &\le -(x^4 + x^2) \\ xu &\le -x^2 - ax^3 = -x(x + ax^2) \end{aligned}$$

- The above condition is accomplished by choosing

$$u = -ax^2 - x.$$

- With this input function u, we obtain

$$\dot{x} = ax^2 - x^3 + u = -x - x^3$$

which is asymptotically stable. The result is global since V_1 is radially unbounded and $D = \mathbb{R}$.

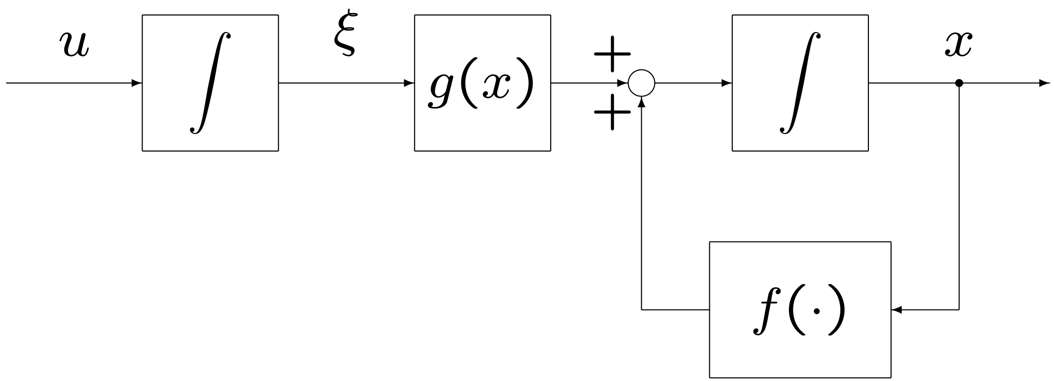
Backstepping

Integrator Backstepping

- Consider a system

$$\begin{aligned}\dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= u.\end{aligned}$$

  Here $x \in \mathbb{R}^n, \xi \in \mathbb{R}$ and $[x, \xi]^T \in \mathbb{R}^{n+1}$
- The function $u \in \mathbb{R}$ is the control input and the functions $f, g : D \to \mathbb{R}^n$ are assumed to be smooth.
- It has a cascade connection structure.



Assumptions

- Assumptions
  (i) The function f satisfies $f(0) = 0$. Thus the origin is an equilibrium point of the subsystem $\dot{x} = f(x)$.
  (ii) The first subsystem can be stabilized by a state feedback $\xi = \phi(x)$.
- Condition (ii) is actually as follows. We assume that there exists a state feedback control law of the form

$$\xi = \phi(x), \quad \phi(0) = 0$$

and a Lyapunov function $V_1 : D \to \mathbb{R}^+$ such that

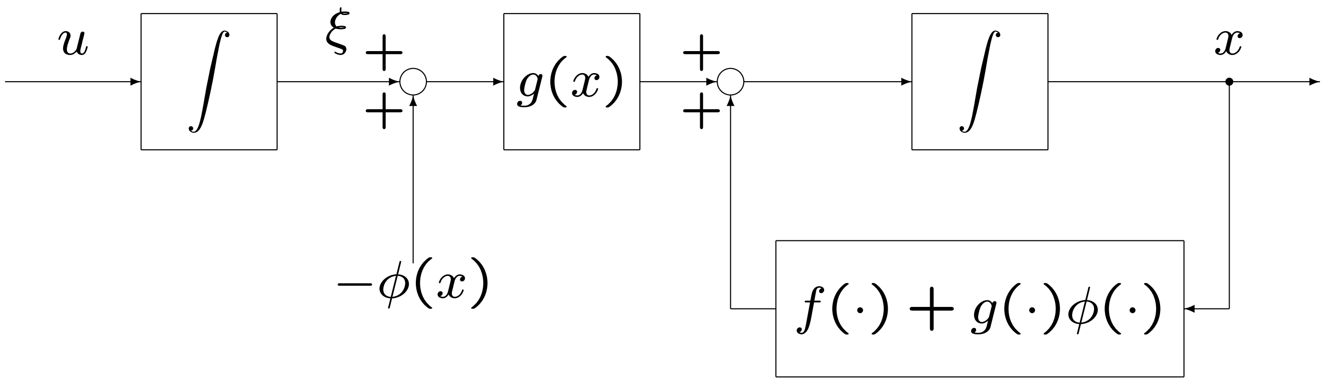
$$\dot{V}_1(x) = \frac{\partial V_1}{\partial x}[f(x) + g(x)\phi(x)] \leq -V_a(x) \leq 0 \quad \forall x \in D$$

where $V_a : D \to \mathbb{R}^+$ is a positive semidefinite function in D.

Backstepping

- An equivalent system

$$\begin{aligned}\dot{x} &= f(x) + g(x)\phi(x) + g(x)(\xi - \phi(x)) \\ \dot{\xi} &= u.\end{aligned}$$



Backstepping

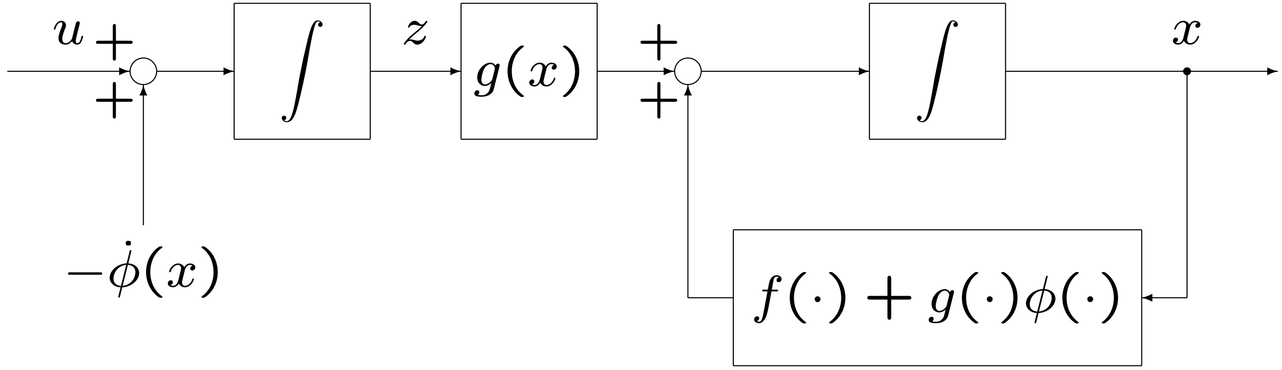
- Define

$$\begin{aligned} z &= \xi - \phi(x) \\ \dot{z} &= \dot{\xi} - \dot{\phi}(x) = u - \dot{\phi}(x) \end{aligned}$$

  where

$$\dot{\phi} = \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}\dot{x}\left(f(x) + g(x)\xi\right)$$

- This change of variables can be seen as **backstepping** $-\phi(x)$ through the integrator.



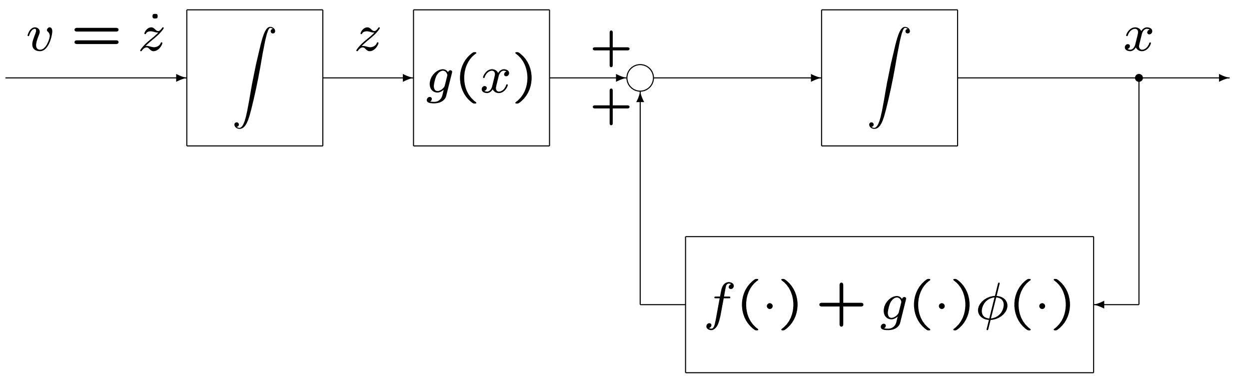
Equivalent System

- Defining $v = \dot{z}$ the resulting system is

$$\begin{aligned} \dot{x} &= f(x) + g(x)\phi(x) + g(x)z \\ \dot{z} &= v. \end{aligned}$$

(i) The system is equivalent to the previous system.

(ii) The system is the cascade connection of two subsystems. However it incorporates the stabilizing state feedbacd $\phi(\cdot)$ and is asymptotically stable when the input is zero.



Stabilization

- To stabilize the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\phi(x) + g(x)z \\ \dot{z} &= v \end{aligned}$$

consider a Lyapunov function candidete of the form

$$V = V(x, \xi) = V_1(x) + \frac{1}{2}z^2.$$

We have

$$\begin{aligned} \dot{V} &= \frac{\partial V_1}{\partial x}(f(x) + g(x)\phi(x) + g(x)z) + z\dot{z} \\ &= \frac{\partial V_1}{\partial x}f(x) + \frac{\partial V_1}{\partial x}g(x)\phi(x) + \frac{\partial V_1}{\partial x}g(x)z + zv. \end{aligned}$$

Stabilization

- We can choose

$$v = -\left(\frac{\partial V_1}{\partial x} g(x) + kz\right), \quad k > 0$$

- Thus

$$\begin{aligned}\dot{V} &= \frac{\partial V_1}{\partial x} f(x) + \frac{\partial V_1}{\partial x} g(x)\phi(x) - kz^2 \\ &= \frac{\partial V_1}{\partial x} (f(x) + g(x)\phi(x)) - kz^2 \\ &\leq -V_a(x) - kz^2\end{aligned}$$

- Now we can conclude that $x = 0, z = 0$ is asymptotically stable.
- Moreover, since $z = \xi - \phi(x)$ and $\phi(0) = 0$ by assumption, the origin of the original system $x = 0, \xi = 0$ is also asymptotically stable.

Stabilization

- If all the conditions hold globally and V_1 is radially unbounded, then the origin is globally asymptotically stable.