System Control Engineering

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Text: Nonlinear Control Systems — Analysis and Design, Wiley
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Today's topics

- Stability notations of nonlinear systems
- Lie derivative
- Lyapunov stability
- Lyapunov's indirect method

Stability notations of nonlinear systems

Stability

- Consider the autonomous system

$$\dot{x} = f(x) \qquad f : D \to \mathbb{R}^n$$

where D is an open and connected subset of $\mathbb{R}^n$.

- Assume that $x = x_e$ is an equilibrium point, i.e., $f(x_e) = 0$.
- **Definition 3.1** The equilibrium point $x = x_e$ is said to be stable if for each $\epsilon > 0$, $\exists \delta(\epsilon) > 0$

$$\|x(0) - x_e\| < \delta \quad \Rightarrow \quad \|x(t) - x_e\| < \epsilon \quad \forall t \geq t_0$$

otherwise, the equilibrium point is said to be unstable.

- This is the weakest form of stability.

Convergent, Asymptotical Stability

- **Definition 3.2** The equilibrium point $x = x_e$ is said to be convergent if for any given $\epsilon_1 > 0$, $\exists \delta_1(\epsilon_1) > 0$ and $\exists T$ such that

$$\|x(0) - x_e\| < \delta_1 \quad \Rightarrow \quad \|x(t) - x_e\| < \epsilon_1 \quad \forall t \geq t_0 + T.$$

- For a convergent equilibrium point we can say

$$\lim_{t \to \infty} x(t) = x_e.$$

- **Definition 3.3** The equilibrium point $x = x_e$ is said to be asymptotically stable if it is both stable and convergent.

Exponential Stability

- **Definition 3.4** The equilibrium point $x = x_e$ is said to be locally exponentially stable if there exist two real constants $\alpha, \lambda > 0$ such that

$$\|x(t) - x_e\| < \alpha \|x(0) - x_e\| e^{-\lambda t} \quad \forall t > 0.$$

- Exponential stability is the strongest form of stability.
- Exponential stability implies asymptotic stability, however, the converse is not true.
- For linear systems they are exponentially stable if it is stable.
- Shift the equilibrium point and hereafter we discuss the stability of the origin.

Lie derivative

Positive Definite Functions

- **Definition 3.5** A function $V : D \to \mathbb{R}$ is said to be **positive semi definite** in D if it satisfies the following conditions:

  (i) $0 \in D$ and $V(0) = 0$.

  (ii) $V(x) \geq 0, \quad \forall x$ in $D - \{0\}$

  $V : D \to \mathbb{R}$ is said to be **positive definite** in D if condition (ii) is replaced by (ii').

  (ii') $V(x) > 0, \quad \forall x$ in $D - \{0\}$

  Finally, $V : D \to \mathbb{R}$ is said to be **negative definite** (semi definite) in D if $-V$ is positive definite (semi definite).

Example

- The simplest and most important class of positive definite function is defined as follows:

$$V(x) = x^T Q x : \mathbb{R}^n \to \mathbb{R}, \quad Q \in \mathbb{R}^{n \times n}, \quad Q = Q^T$$

Since Q is symmetric, we know that its eigenvalues are all real.

$$\begin{aligned}
V \text{ is positive definite } &\Leftrightarrow \lambda_i > 0, \forall i = 1, \ldots, n \\
V \text{ is positive semi definite } &\Leftrightarrow \lambda_i \geq 0, \forall i = 1, \ldots, n \\
V \text{ is negative definite } &\Leftrightarrow \lambda_i < 0, \forall i = 1, \ldots, n \\
V \text{ is positive semi definite } &\Leftrightarrow \lambda_i \leq 0, \forall i = 1, \ldots, n
\end{aligned}$$

Thus for example:

$$V(x) = a x_1^2 + b x_2^2 = [\, x_1 \quad x_2 \,] \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0, \quad \forall a, b > 0.$$

Positive definite functions (PDFs)

- PDFs constitute the basic building block of the Lyapunov theory.
- PDFs can be seen as an abstraction of the total **energy** stored in the system.
- All of the Lyapunov stability theorems focus on the study of the time derivative of a positive definite function along the trajectory of $\dot{x} = f(x)$.

Time Derivative along the Trajectory

- Time derivative of V along the trajectory:
  1. Trajectory

$$\dot{x} = f(x)$$

  2. Time derivative

$$\begin{aligned}
\dot{V}(x) &= \frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \nabla V f(x) \\
&= \left[\, \frac{\partial V}{\partial x_1} \quad \frac{\partial V}{\partial x_2} \quad \cdots \quad \frac{\partial V}{\partial x_n} \,\right] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}
\end{aligned}$$

Lie derivative

- **Definition 3.6** Let $V : D \to \mathbb{R}$ and $f : D \to \mathbb{R}^n$. The **Lie derivative** of V along f, denoted by $L_f V$, is defined by

$$L_f V(x) = \frac{\partial V}{\partial x} f(x).$$

Thus according to this definition, we have that

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = \nabla V f(x) = L_f V(x).$$

Example: Lie derivative

- Example: Let

$$\dot{x} = \begin{bmatrix} ax_1 \\ bx_2 + \cos x_1 \end{bmatrix}$$

and define $V = x_1^2 + x_2^2$. Thus we have

$$\begin{aligned} \dot{V}(x) &= L_f V(x) = [\,2x_1 \quad 2x_2\,] \begin{bmatrix} ax_1 \\ bx_2 + \cos x_1 \end{bmatrix} \\ &= 2ax_1^2 + 2bx_2^2 + 2x_2 \cos x_1. \end{aligned}$$

- It is clear from this example that the $\dot{V}(x)$ depends on the system's equation $f(x)$ and thus it will be different for different systems, even if V is the same.

Lyapunov stability

Lyapunov Stability Theorem

- **Theorem 3.1** Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, $f : D \to \mathbb{R}^n$, and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

  (i) $V(0) = 0$,

  (ii) $V(x) > 0 \quad \text{in} \quad D - \{0\}$

  (iii) $\dot{V}(x) \leq 0 \quad \text{in} \quad D - \{0\}$,

  then $x = 0$ is stable.
- The theorem implies that a **sufficient condition** for the stability of the equilibrium point $x = 0$ is that there exists a continuously differentiable positive definite function $V(x)$ such that $\dot{V}(x)$ is negative semi definite in a neighborhood of $x = 0$.

Lyapunov's Asymptotic Stability Theorem

- **Theorem 3.2** Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, $f : D \to \mathbb{R}^n$, and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

  (i) $V(0) = 0$,

  (ii) $V(x) > 0 \quad \text{in } D - \{0\}$

  (iii) $\dot{V}(x) < 0 \quad \text{in } D - \{0\}$,

  thus $x = 0$ is asymptotically stable.

Proof of theorem 3.1

- Choose $r > 0$ such that the closed ball

$$B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$$

is contained in D. Let

$$\alpha = \min_{\|x\|=r} V(x).$$

Now choose $\beta \in (0, \alpha)$ and denote

$$\Omega_\beta = \{x \in B_r : V(x) \leq \beta\}.$$

Thus, by construction, $\Omega_\beta \subset B_r$. Now suppose that $x(0) \in \Omega_\beta$. By assumption (iii) of the theorem we have that

$$\dot{V}(x) \leq 0 \quad \Rightarrow \quad V(x) \leq V(x(0)) \leq \beta \quad \forall t \geq 0.$$

Proof of theorem 3.1 (Cont.)

- It then follows that any trajectory starting in Ω_β at $t = 0$ stays inside Ω_β for all $t \geq 0$. Moreover, by the continuity of $V(x)$ it follows that $\exists \delta > 0$ such that

$$\|x\| < \delta \quad \Rightarrow \quad V(x) < \beta \quad (B_\delta \in \Omega_\beta \in B_r).$$

Thus we have

$$\|x(0)\| < \delta \quad \Rightarrow \quad x(t) \in \Omega_\beta \in B_r \quad \forall t > 0$$

and then

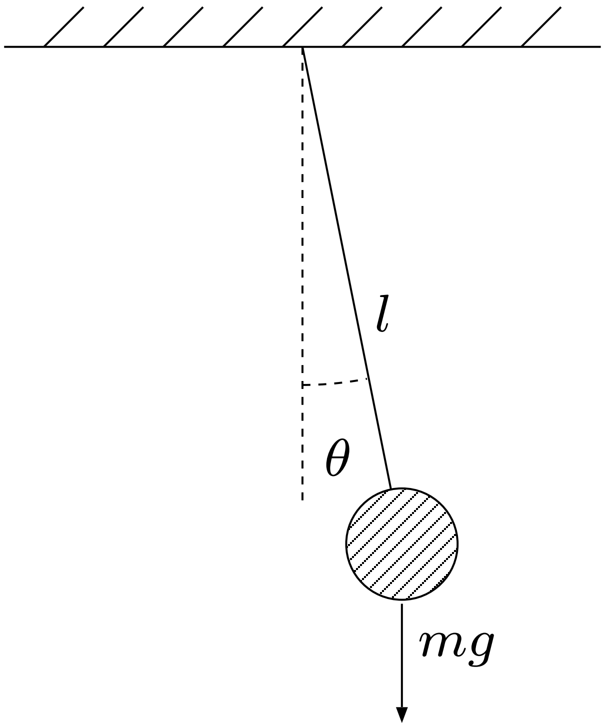
$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < r \leq \epsilon \quad \forall t \geq 0$$

which means that the equilibrium $x = 0$ is stable.

Lyapunov function

- Finding a positive definite function is easy because V is independent of the dynamics of the differential equation under study.
- While $\dot{V}$ depends on this dynamics.
- For this reason, when a function V is proposed as possible candidate to prove the stability, V is said to be a **Lyapunov function candidate**.
- If in addition $\dot{V}$ happens to be negative definite, then V is said to be a **Lyapunov function** for that particular equilibrium point.

Example: Pendulum without friction



- Dynamical equation:

$$m\ell\ddot{\theta} + mg\sin\theta = 0$$

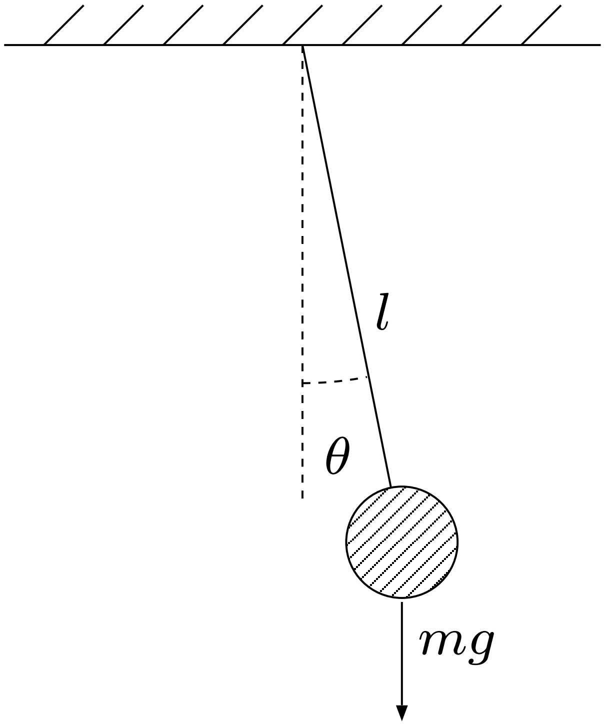
- State variables: $x_1 = \theta, x_2 = \dot{\theta}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 \end{aligned}$$

- Total Energy of the system

$$\begin{aligned} E &= K + P = \frac{1}{2}m(\omega\ell)^2 + mgh \\ &= \frac{1}{2}m\ell^2x_2^2 + mgl(1-\cos x_1) \end{aligned}$$

Example (Cont.)



- Define:

$$V(x) = E = \frac{1}{2}m\ell^2 x_2^2 + mgl(1 - \cos x_1)$$

- Clearly $V(0) = 0$, however, we have $V(x) = 0$ whenever $x = [x_1, x_2]^T = [2k\pi, 0]^T$. Thus V is not positive definite.
- Restrict the domain:

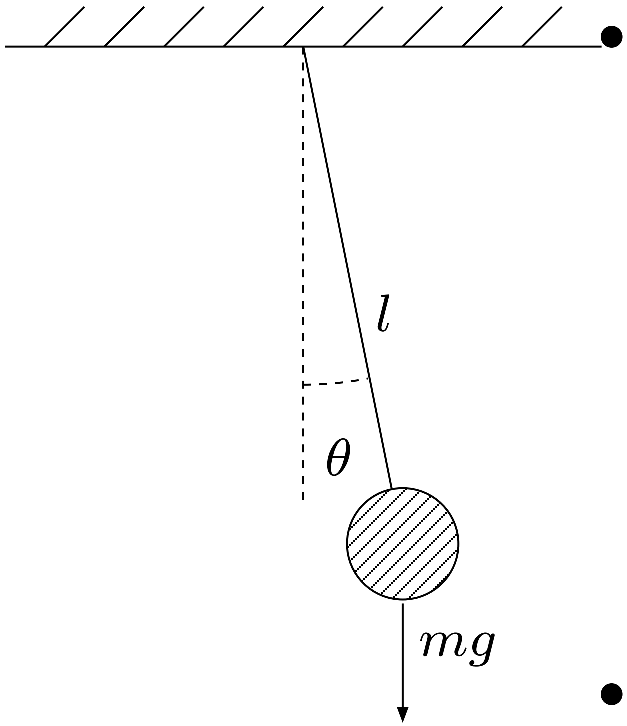
$$x_1 \in (-2\pi, 2\pi),$$

i.e.,

$$V : D \to \mathbb{R}, \quad D = [(-2\pi, 2\pi), \mathbb{R}]^T$$

- With this restriction, V is positive definite.

Example (Cont.)



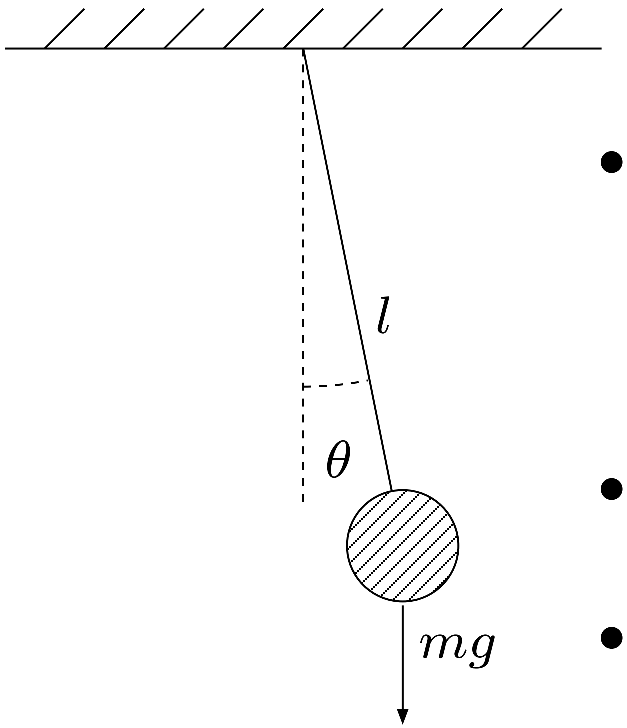
- Evaluate the derivative

$$V(x) = \frac{1}{2} m\ell^2 x_2^2 + mgl(1 - \cos x_1)$$

$$\begin{aligned} \dot{V}(x) &= \nabla V f(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \\ &= \begin{bmatrix} mg\ell \sin x_1 & m\ell^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{\ell} \sin x_1 \end{bmatrix} \\ &= mg\ell x_2 \sin x_1 - mg\ell x_2 \sin x_1 = 0 \end{aligned}$$

- Thus $\dot{V}(x) = 0$ and the origin is stable.

Example: Pendulum with friction



- Dynamical equation:

$$m\ell\ddot{\theta} + mg\sin\theta + k\ell\dot{\theta} = 0$$

- State variables: $x_1 = \theta, x_2 = \dot{\theta}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

- $x = [x_1, x_2]^T = [0, 0]^T$ is an equilibrium point.
- Total Energy:

$$V(x) = \frac{1}{2}m\ell^2 x_2^2 + mgl(1 - \cos x_1)$$

Example (Cont.)

- Evaluate the derivative

$$V(x) = \frac{1}{2}m\ell^2 x_2^2 + mgl(1 - \cos x_1)$$

$$\begin{aligned} \dot{V}(x) &= \begin{bmatrix} mg\ell \sin x_1 & m\ell^2 x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix} \\ &= -k\ell^2 x_2^2 \end{aligned}$$

- $\dot{V}(x)$ is negative semi-definite.
- The origin is stable but cannot conclude asymptotic stability.
- The result is disappointing since we know that it is asymptotically stable.
- The Lyapunov theorem is **sufficient** condition.

LaSalle's Asymptotic Stability Theorem

- **Theorem 3.6** Let $x = 0$ be an equilibrium point of

$$\dot{x} = f(x), \quad f : D \to \mathbb{R}^n,$$

  and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

  (i) $V(0) = 0$,

  (ii) $V(x) > 0$ in D, where we assume that $0 \in D$

  (iii) $\dot{V}(x) \leq 0$ in a bounded region $R \subset D$

  (iv) $\dot{V}(x)$ does not vanish identically along any trajectory in R other than $x = 0$.

  then $x = 0$ is asymptotically stable.

Example

- For the pendulum with friction, we know

$$V(x) = \frac{1}{2}m\ell^2 x_2^2 + mgl(1 - \cos x_1)$$
$$\dot{V}(x) = -k\ell^2 x_2^2$$

- $\dot{V}(x)$ is negative semi-definite in $D = [(-\pi, \pi), \mathbb{R}]^T$.
- Suppose a closed region

$$R = [(-\pi, \pi), (-a, a)]^T \qquad \text{for any } a > 0.$$

- Check the condition (iv).

$$\dot{V} = 0 \quad \Rightarrow \quad 0 = -k\ell^2 x_2^2 \quad \Leftrightarrow \quad x_2 = 0$$

thus, $x_2 = 0, \forall t$. This also conclude that $\dot{x}_2 = 0$.

Example (Cont.)

- State equation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

- $x_2 = 0$ and $\dot{x}_2 = 0$, thus $\sin x_1 = 0$.
- Restricting $x_1 \in (-\pi, \pi)$, $\sin x_1 = 0$ if and only if $x_1 = 0$.
- It follows that $\dot{V}(x)$ does not vanish identically along any solution other than $x = 0$, and the origin is locally asymptotically stable.