System Control Engineering

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Text: Nonlinear Control Systems — Analysis and Design, Wiley
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Web: `http://www.ic.is.tohoku.ac.jp/~koichi/system_control/`

- Nonlinear systems expression
  - Unforced, Autonomous, Example
- Equilibrium points
  - Examples – first order, second order
- Phase-plane analysis
  - Vector field diagram
  - Examples
  - Limit cycle – Van der Pol oscillator, MATLAB
  - Lorenz attractor – MATLAB
- Exercises

Nonlinear Systems

- State $x(t)$, Input $u(t)$, Output $y(t)$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix}$$

- **System**: A set of first-order ordinary differential equations

$$\begin{aligned} \dot{x}(t) &= f(x(t), t, u(t)) \\ y(t) &= h(x(t), t, u(t)) \end{aligned}$$

i.e., (hereafter, (t) may be omitted)

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n, t, u_1, \ldots, u_p) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n, t, u_1, \ldots, u_p) \\ y_1 &= h_1(x_1, \ldots, x_n, t, u_1, \ldots, u_p) \\ &\vdots \\ y_m &= h_m(x_1, \ldots, x_n, t, u_1, \ldots, u_p) \end{aligned}$$

Examples of Nonlinear Systems

- **Unforced system**: Input u is identically zero, i.e., $u(t) = 0$

$$\dot{x} = f(x, t, 0) = f(x, t)$$

- **Autonomous system**: $f(x, t)$ is not a function of time

$$\dot{x} = f(x)$$

  Autonomous systems are invariant to shifts in the time origin, i.e., changing the time variable from t to $\tau = t - \alpha$ does not change the right-hand side of the equation.

Example

- Input: u, Output: x
- Consider a system:

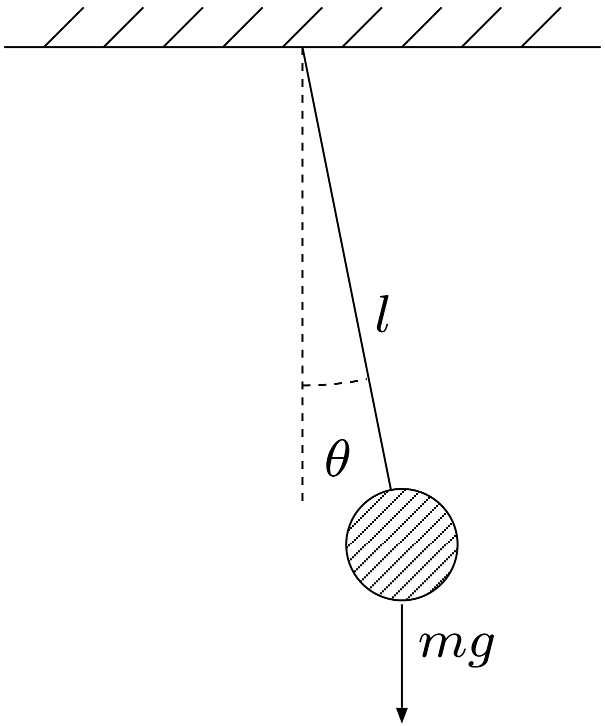
$$m\ddot{x} + d(x, \dot{x}) + k(x) = u$$

- $x_1 = x, x_2 = \dot{x}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{d(x_1, x_2)}{m} - \frac{k(x_1)}{m} + \frac{1}{m}u \end{aligned}$$

i.e., the left-hand side should be first derivative of x, the right-hand side should not contain $\dot{x}$.

Example



- Dynamical equation:

$$ml\ddot{\theta} + kl\dot{\theta} + mg\sin\theta = 0$$

- State variables: $x_1 = \theta, x_2 = \dot{\theta}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

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Equilibrium Points

- **Definition**: A point $x = x_e$ is said to be an **equilibrium point** of the autonomous system

$$\dot{x} = f(x)$$

if it has the property that whenever the state of the system starts at x_e, it remains at x_e for all future time

$$x(t_0) = x_e \quad \Rightarrow \quad x(t) \equiv x_e, \ \forall t \geq t_0,$$

i.e.,

$$\dot{x} = 0.$$

- **Property**: The equilibrium points are the real roots of the equation $f(x_e) = 0$.

Example of Equilibrium Points

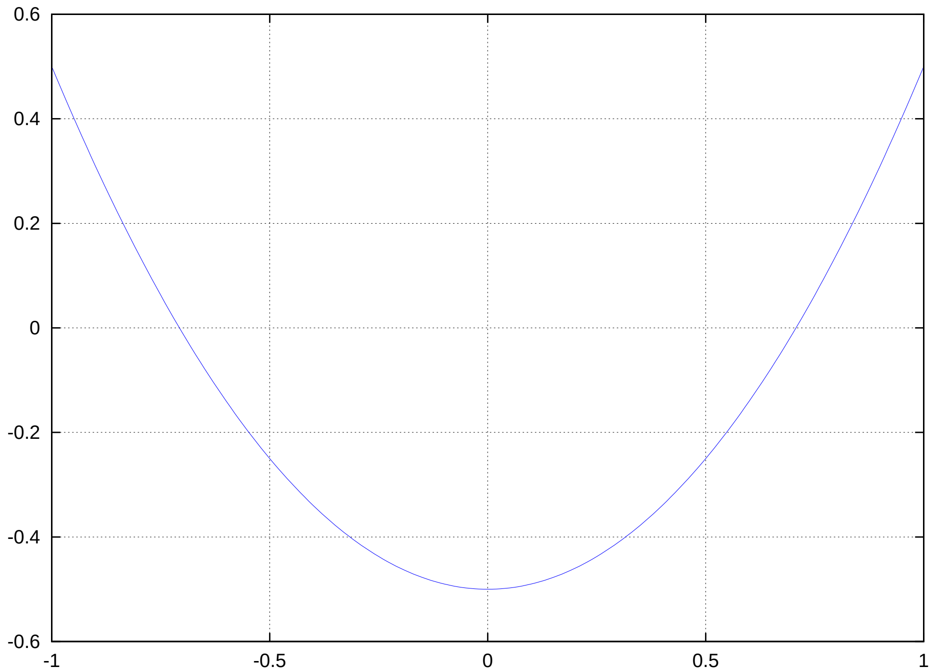
- Consider the following system where r is a parameter.

$$\dot{x} = -r + x^2$$

1. If $r > 0$, the system has two equilibrium points $x = \pm\sqrt{r}$.
2. If $r = 0$, both of the equilibrium points collapse, the equilibrium point is $x = 0$.
3. If $r < 0$, then the system has no equilibrium points.

$$\dot{x} = -r + x^2 \quad (r > 0)$$

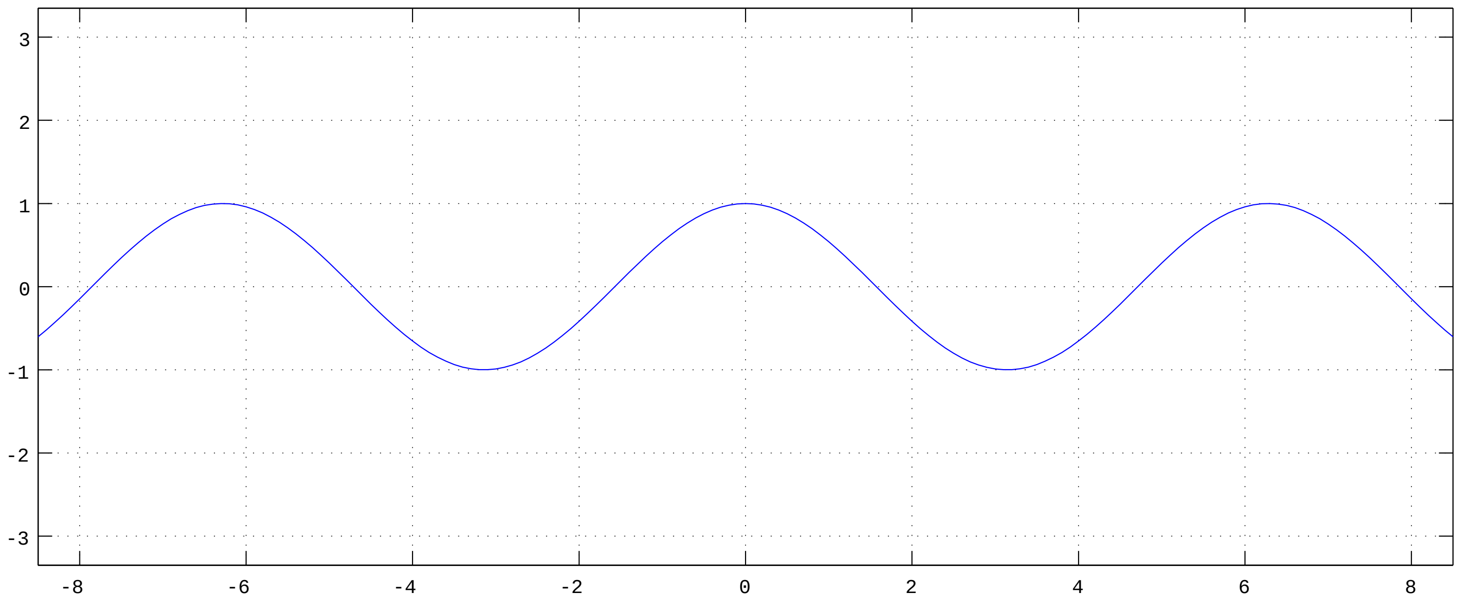
When $\dot{x} > 0$, the trajectories move to the right, and vice versa. Thus $x_e = -\sqrt{r}$ is attractive, $x_e = \sqrt{r}$ is repelling.



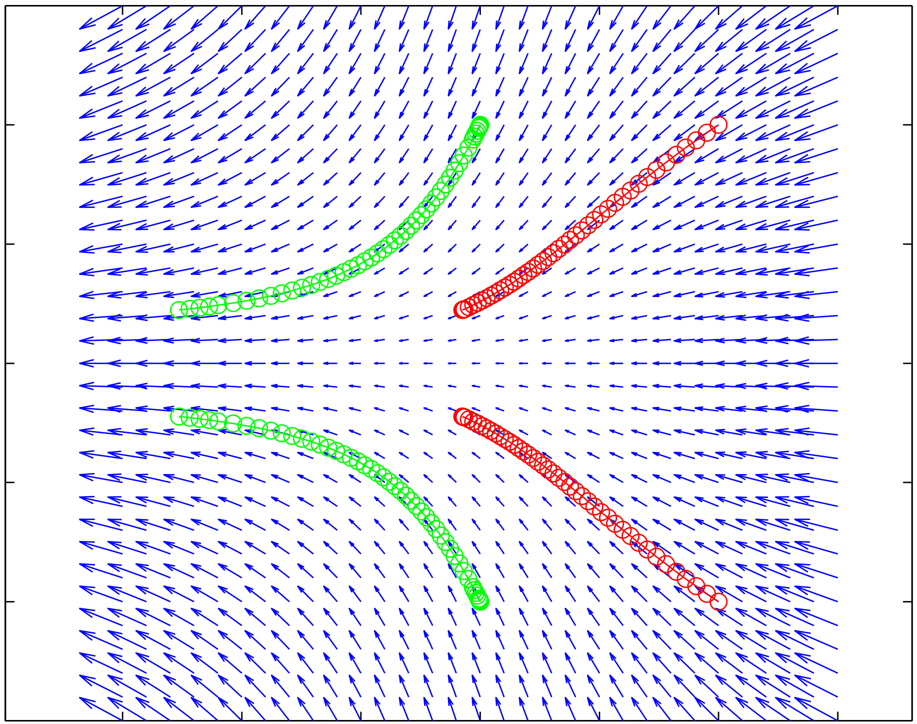
$$r = 0.5$$

$$\dot{x} = \cos x$$

where $\dot{x} = 0$ are equilibrium points.

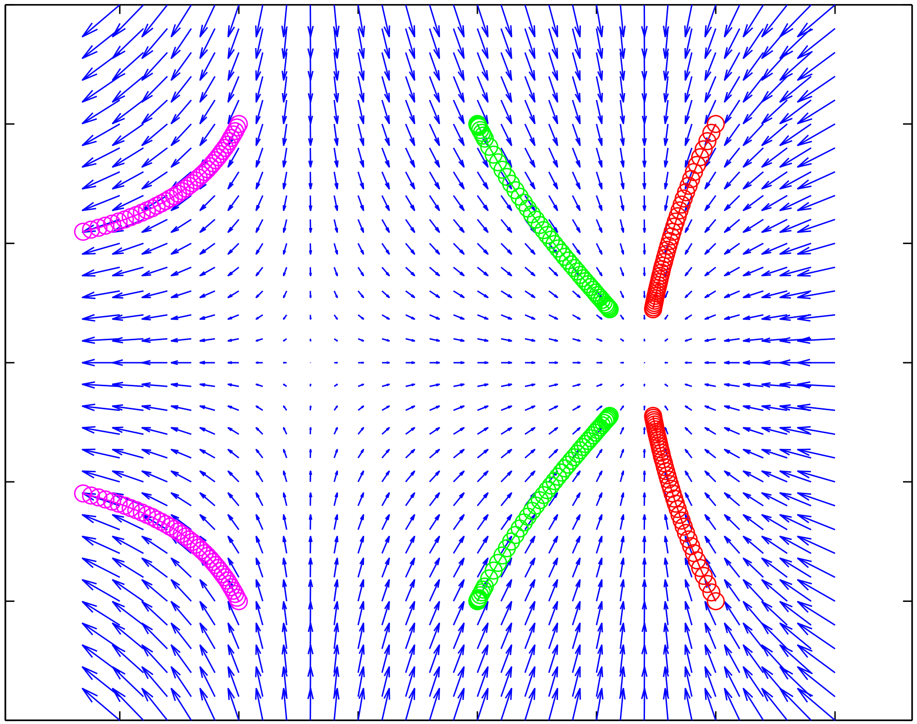


$$\begin{aligned} \dot{x}_1 &= r - x_1^2 \quad (r = 0.5) \\ \dot{x}_2 &= -x_2 \end{aligned}$$



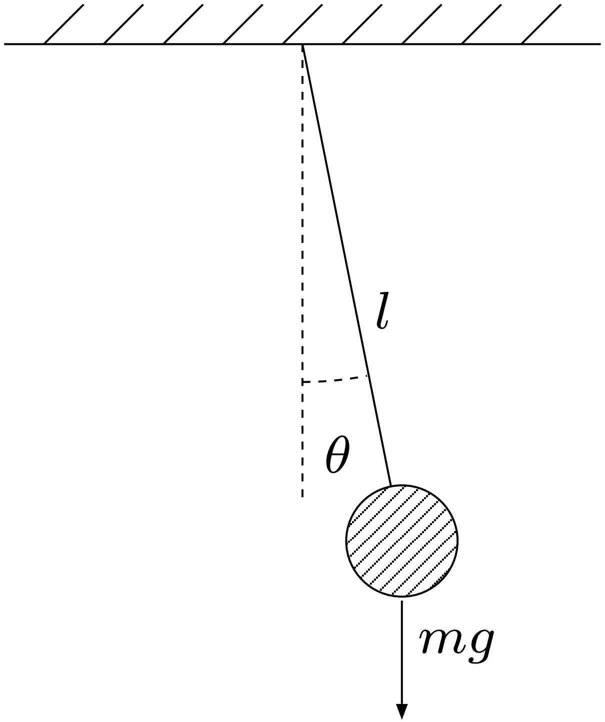
$$\dot{x}_1 = r - x_1^2 \quad (r = 0.5)$$
$$\dot{x}_2 = -x_2$$

where $(\sqrt{r}, 0)$ is stable, $(-\sqrt{r}, 0)$ is unstable.



```
[x, y]= meshgrid(-1.5:0.1:1.5, -1.5:0.1:1.5);
global r;
r = 0.5;
px=r-x.*x;
py=-y;
quiver(x,y,px,py,1.5);
[t,xx]=ode45(@func_bifur,[0 1.5],[0;1]);
plot(xx(:,1),xx(:,2),'go-');
-----
function dx = func_bifur(t, x)
global r;
dx = [r-x(1)*x(1); -x(2)];
end
```

Pendulum



- Dynamical equation:

$$ml\ddot{\theta} + kl\dot{\theta} + mg\sin\theta = 0$$

- State variables: $x_1 = \theta, x_2 = \dot{\theta}$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

- Equilibrium points

$$\begin{aligned}0 &= x_2 \\ 0 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \\ (x_1, x_2) &= (n\pi, 0), \quad n = 0, \pm 1, \ldots\end{aligned}$$

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- Consider the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

- The solution of the differential equation with an initial condition $x_0 = [x_{10}, x_{20}]$ is called a **trajectory** from x_0.
- The trajectory in x_1–x_2 plane is called **phase-plane**.
- $f(x)$ in

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = f(x)$$

is called a **vector field**.

Vector Field Diagram

- To each point x^* in the plane we can assign a vector with amplitude and direction of $f(x^*)$.
- For easy visualization we can represent $f(x)$ as a vector based at x, i.e., we assign to x the directed line segment from x to $x + f(x)$.
- Repeating this operation at every point in the plane, we obtain a **vector field diagram**.

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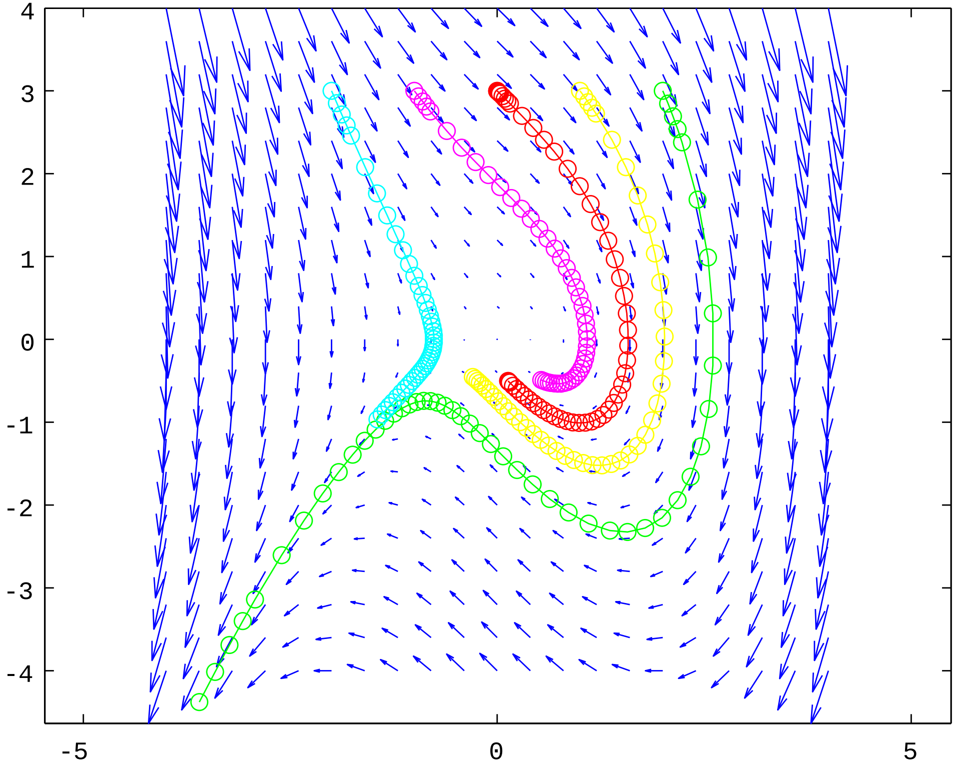
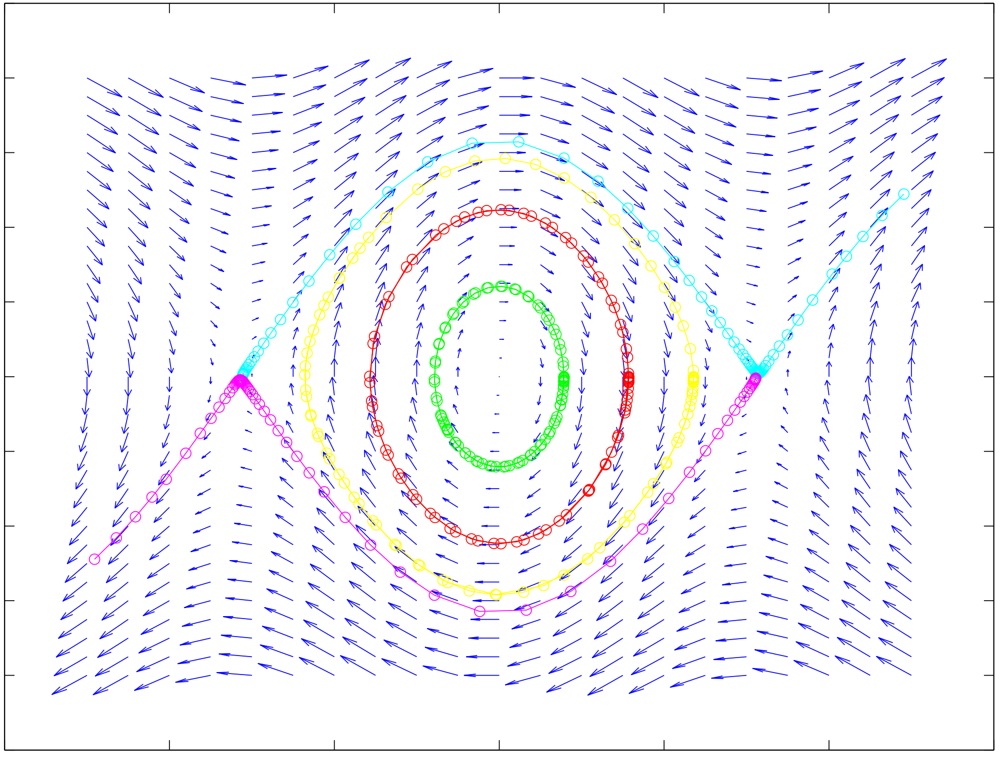
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^2 - x_2$$



$(0 \le t \le 3)$

Vector Field Diagram of Pendulum

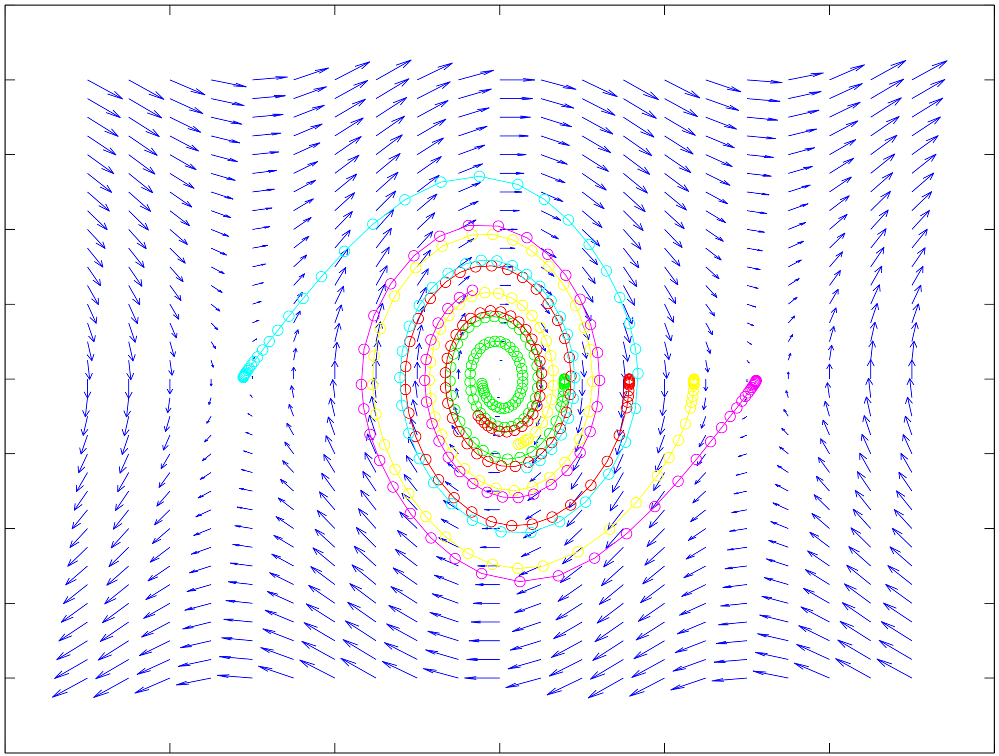
$$\begin{aligned}\dot{x}_1 &= x_2 \qquad\qquad (g = 10, l = 1)\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 \qquad (0 \le t \le 5)\end{aligned}$$



$(\pi/4, 0), (\pi/2, 0), (3\pi/4, 0), (-\pi - \epsilon, 0.04), (\pi + \epsilon, -0.04)$

Pendulum with Friction

$$\begin{aligned} \dot{x}_1 &= x_2 & (g = 10, l = 1, k = 0.5, m = 1) \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 & (0 \le t \le 5) \end{aligned}$$



$(\pi/4, 0), (\pi/2, 0), (3\pi/4, 0), (-\pi - \epsilon, 0.04), (\pi + \epsilon, -0.04)$

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Limit Cycles

- **Oscillations**: Characteristics of higher-order nonlinear systems
- A system oscillates when it has a **nontrivial** periodic solution (not the one found in LTI imaginary case).

$$\exists t_0 > 0, \quad \forall t \geq t_0, \quad x(t+T) = x(t)$$

- Stable, self-excited oscillations: **limit cycles**.

Example of Limit Cycles (Van der Pol)

- Consider the following system:

$$\ddot{y} - \mu(1 - y^2)\dot{y} + y = 0 \qquad \mu > 0$$

- Define $x_1 = y$, and $x_2 = \dot{y}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \mu(1 - x_1^2)x_2 \end{aligned}$$

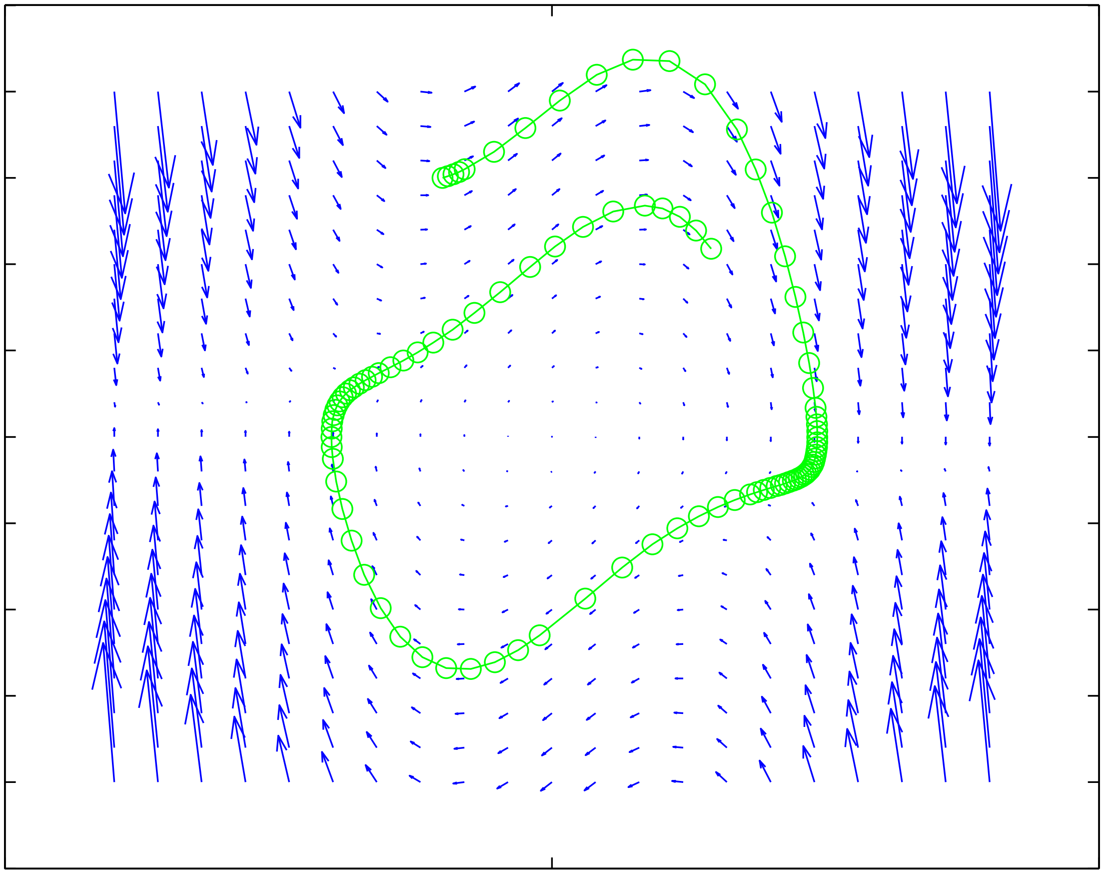
- Note that if $\mu = 0$, the resulting system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

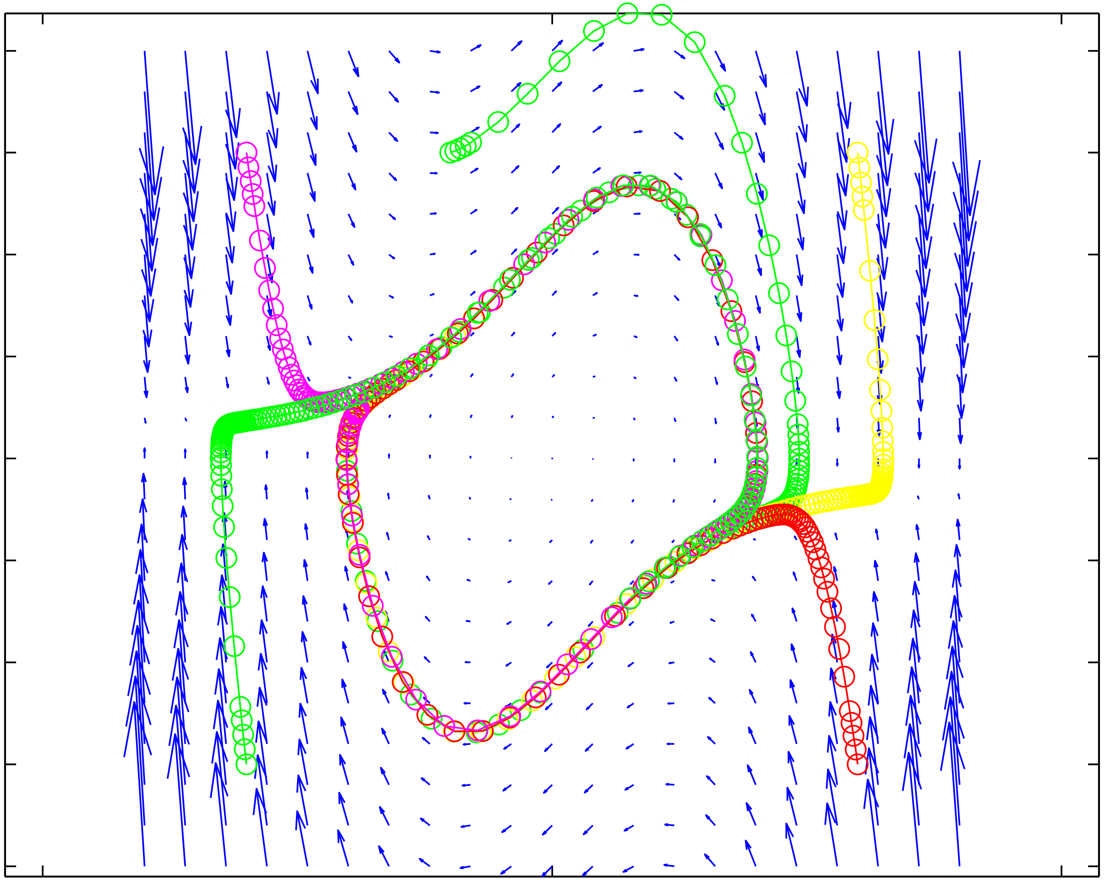
which is LTI and has circular trajectories.

Example of Limit Cycles (Van der Pol)

$$\dot{x}_1 = x_2 \qquad (0 \le t \le 8)$$
$$\dot{x}_2 = -x_1 + \mu(1 - x_1^2)x_2 \qquad (\mu = 1)$$

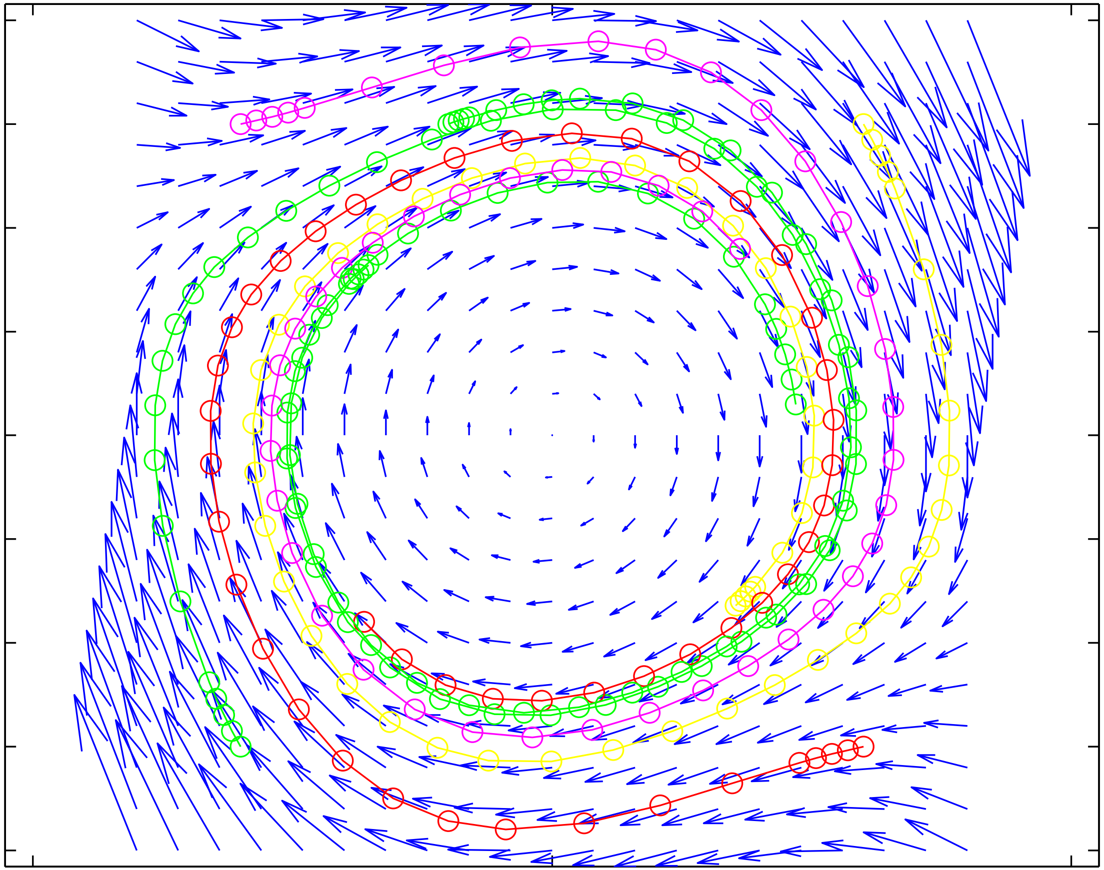


Example of Limit Cycles (Van der Pol)



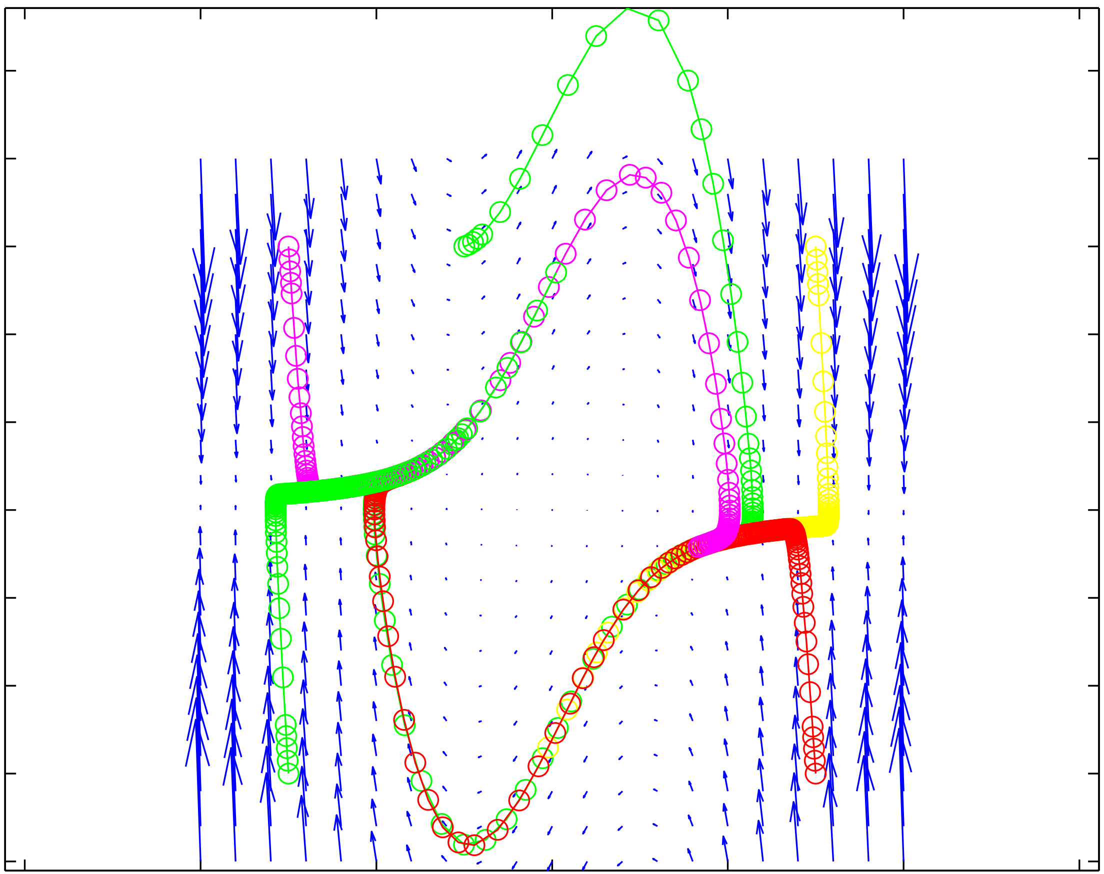
$\mu = 1$, (-3,3), (-1,3), (3,3), (3,-3), (-3,3)

Example of Limit Cycles (Van der Pol)



$\mu = 0.1$, (-3,3), (-1,3), (3,3), (3,-3), (-3,3)

Example of Limit Cycles (Van der Pol)



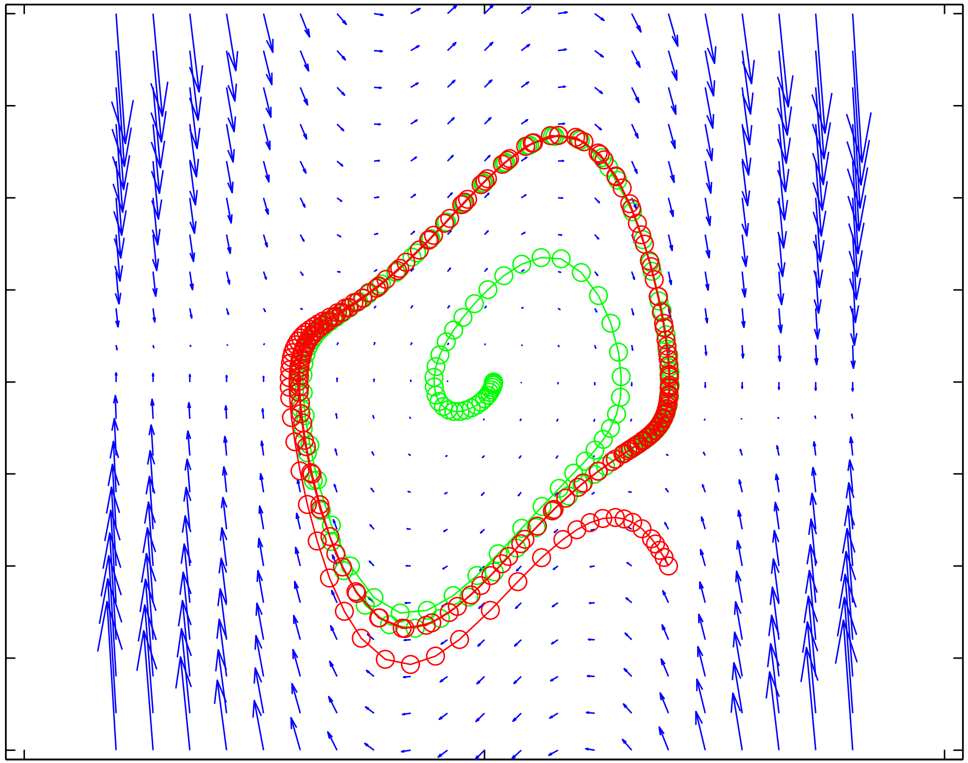
$\mu = 2$, (-3,3), (-1,3), (3,3), (3,-3), (-3,3)

Example of Limit Cycles (Van der Pol)

```
[x, y]= meshgrid(-4:0.4:4, -4:0.4:4);
global mu;  mu=1;
px=y;
py=-x+mu*(1-x.*x).*y;
quiver(x,y,px,py,3);
[t,xx]=ode45(@vdp,[0 8],[-1;3]);
plot(xx(:,1),xx(:,2),'go-');
-----
function dx=vdp(t,x)
global mu;
dx=[x(2); -x(1)+mu*(1-x(1)*x(1))*x(2)];
end
```

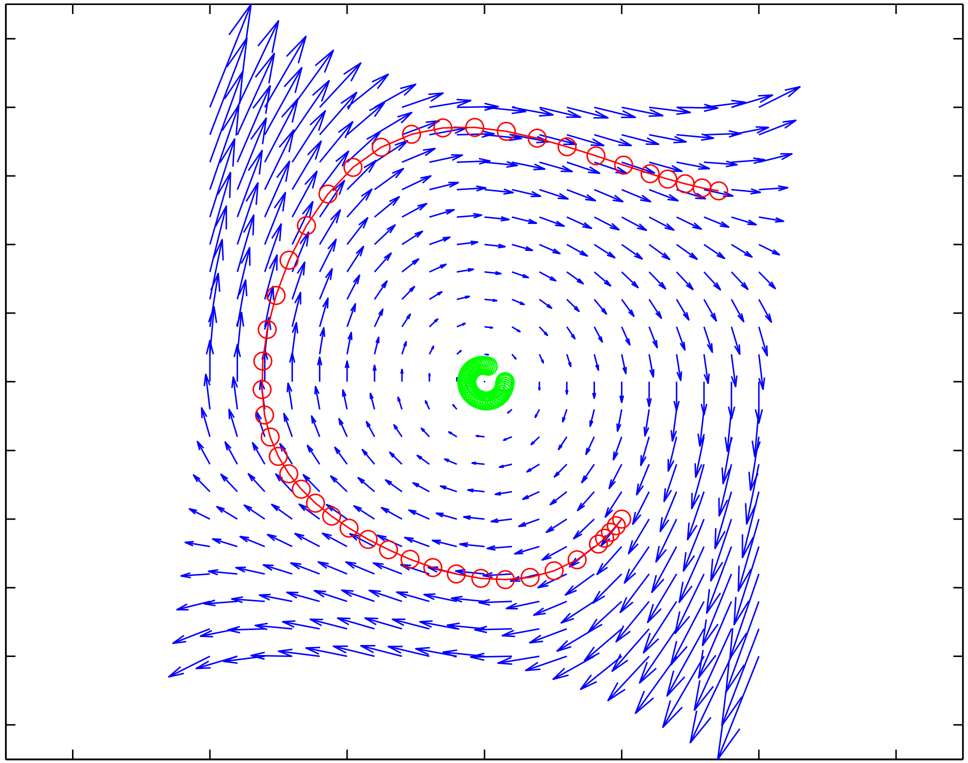
Van der Pol Limit Cycle

- There is only one isolated orbit (**Limit Cycle**).
- All trajectories converge to this trajectory as $t \to \infty$, i.e., it is a **stable limit cycle**.



$\mu = 1$, (0.1,0.0), (2,-2)

Van der Pol Limit Cycle (Unstable)



$\mu = -0.1$, (0.3,0.0), (2,-2)

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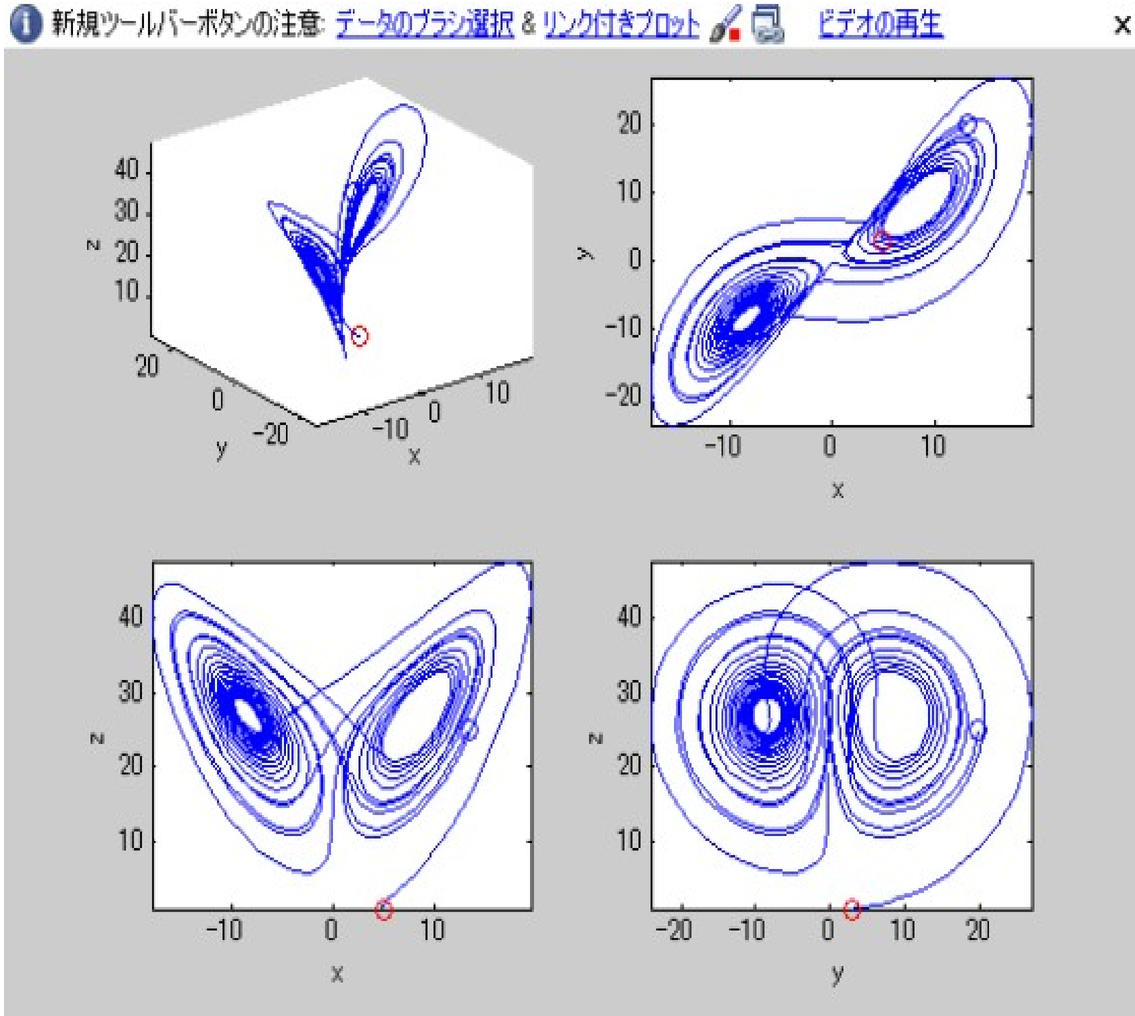
- **Chaos**: Consider the following system of nonlinear equations (Ed Lorenz, 1963)

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

where $\sigma, r, b > 0$.

Lorenz Attractor



Example of Limit Cycles (Lorenz attractor)

```
r = 2;
c = 2;
[t,xyz] = ode45('lorenz',[0,30],[5;3;1]);
x = xyz(:,1); xmin = min(x); xmax = max(x);
y = xyz(:,2); ymin = min(y); ymax = max(y);
z = xyz(:,3); zmin = min(z); zmax = max(z);
plot3(x(1:i),y(1:i),z(1:i),...
      x(1),y(1),z(1),'or',...
      x(i),y(i),z(i),'ob');
axis([xmin xmax ymin ymax zmin zmax]);
xlabel('x'); ylabel('y'); zlabel('z');
```

Example of Limit Cycles (Lorenz attractor)

```
function xyz = lorenz(t,y)
   s = 10;   b = 8/3;   r = 28;
   xyz = [ -s .* y(1) + s .* y(2)
            r .* y(1) - y(2) - y(1) .* y(3)
            y(1) .* y(2) - b .* y(3) ];
end
```

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- For each of the following systems, (i) find the equilibrium points, (ii) plot the phase portrait, and (iii) classify each equilibrium point as stable or unstable.

$$(a) \quad \begin{cases} \dot{x}_1 = x_1 - x_1^3 + x_2 \\ \dot{x}_2 = -x_2 \end{cases}$$

$$(b) \quad \begin{cases} \dot{x}_1 = -x_2 + 2x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = x_1 + 2x_2(x_1^2 + x_2^2) \end{cases}$$

$$(c) \quad \begin{cases} \dot{x}_1 = \cos x_2 \\ \dot{x}_2 = \sin x_1 \end{cases}$$