Linear Time-Invariant Systems

- Linear time-invariant (LTI) systems
  - Expression, Definition, Behavior
- Phase-plane analysis
  - Vector field diagram
  - Examples (Real eigenvalues)
  - Diagonalization
  - MATLAB codes
  - Examples (Real eigenvalues)
  - Examples (Duplicate eigenvalues)
  - Examples (Complex eigenvalues)
- Nonlinear systems

Linear and Nonlinear

- Linear: $y = ax$
- Linear?: $y = ax + b$
- Nonlinear: $y = x^2$
- Nonlinear: $y = \sin(x)$

Linear function $f(x)$

- Additivity: $f(x + y) = f(x) + f(y)$
- Homogeneity: $f(ax) = af(x)$
- Superposition Principle: $f(ax + by) = af(x) + bf(y)$

- **Linear Time Invariant Systems**

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where A, B, C, D are constant matrices.
  - State $x(t)$ expresses the system condition
  - Input $u(t)$ drives the system
  - Output $y(t)$ is the output of the system

- **Modes**:

Autonomous:	$\dot{x}(t) = Ax(t)$	$\rightarrow$	$x(t) = e^{At}x(0)$
Drive:	$\dot{x}(t) = Bu(t)$	$\rightarrow$	$x(t) = \int Bu(t)dt$

and C, D are the coefficients of these modes.

Linear Time-Invariant Systems

Suppose an input $u_1(t)$ generates an output $y_1(t)$, also an input $u_2(t)$ generates an output $y_2(t)$.

- **Linearity**: The system is **linear** if and only if

$$au_1(t) + bu_2(t) \quad \text{generates} \quad ay_1(t) + by_2(t)$$

- **Time-Invariant**: The system is **time invariant** if and only if, the following is true $\forall \tau$

$$u_2(t) = u_1(t - \tau) \quad \text{generates} \quad y_2(t) = y_1(t - \tau)$$

Behavior of LTI systems

- **Autonomous mode:**

$$\dot{x}(t) = Ax(t), \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- **Trajectory:** The solution $x(t)$ of the differential equation with a initial condition

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

is called a **trajectory** from $x(0)$.

- **Phase Plane:** The trajectory expressed in x_1–x_2 plane.

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Vector Field Diagram

- To each point x in the phase plane we can assign a vector with amplitude and direction of

$$\dot{x} = Ax.$$

- For easy visualization we can represent $\dot{x}$ as a vector based at x, i.e., we assign to x the directed line segment from x to $x + Ax$.
- Repeating this operation at every point in the plane, we obtain a **vector field diagram**.

LTI Systems: Example

- Consider the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Diagonalize:

$$\begin{bmatrix} \dot{x}_1(t) + 3\dot{x}_2(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) + 3x_2(t) \\ x_2(t) \end{bmatrix}$$

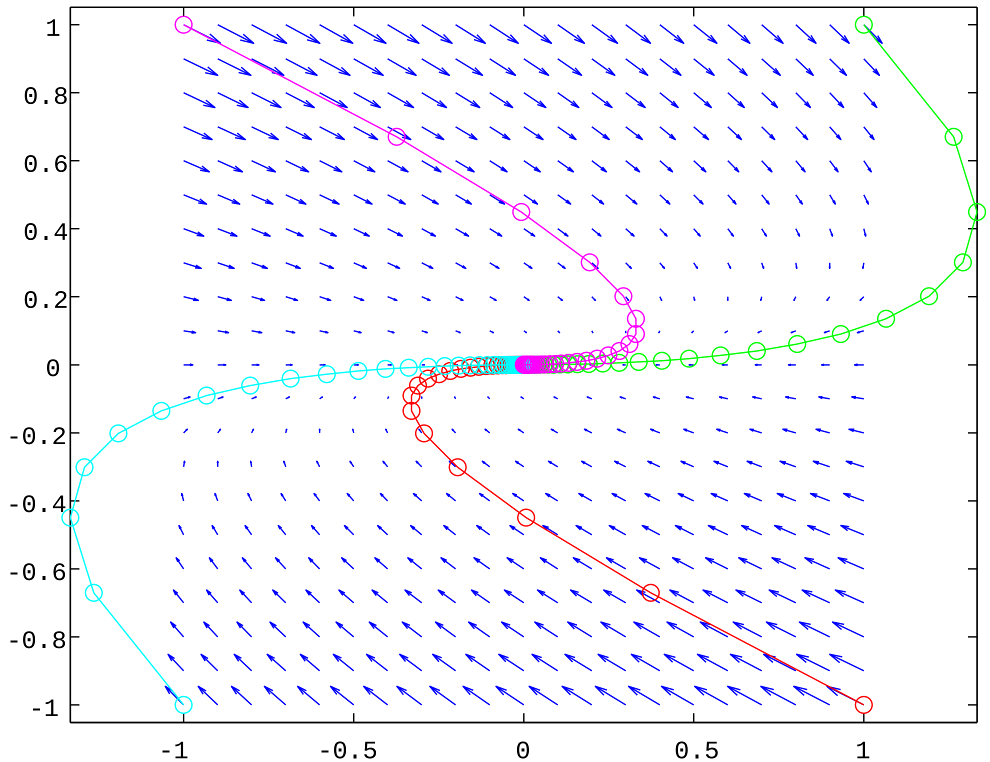
- Let $z_1(t) = x_1(t) + 3x_2(t)$ and $z_2(t) = x_1(t)$, then we have

$$\begin{aligned} \dot{z}_1(t) &= -z_1(t) \\ \dot{z}_2(t) &= -2z_2(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} z_1(t) &= z_1(0)e^{-t} \\ z_2(t) &= z_2(0)e^{-2t} \end{aligned}$$

Phase-Plane Analysis of LTI Systems

- The trajectory from $(x_1(0), x_2(0))$ is:

$$\begin{aligned} x_1(t) &= x_1(0)e^{-t} + 3x_2(0)(e^{-t} - e^{-2t}) \\ x_2(t) &= x_2(0)e^{-2t} \end{aligned}$$



LTI Systems and Eigenvalues

- Second-order LTI system:

$$\dot{x} = Ax, \quad A \in \mathbb{R}^{2\times 2}$$

where $\mathbb{R}^{2\times 2}$ denotes a set of 2×2 matrices with real entries.

- The solution with initial condition $x(0)$

$$x(t) = e^{At}x(0), \quad e^{At} = I + A + \frac{1}{2}A^2 + \cdots$$

- Eigenvalues λ_1, λ_2, Eigenvectors v_1, v_2:

$$\lambda_1 v_1 = Av_1, \quad \lambda_2 v_2 = Av_2, \quad TD = AT$$

where

$$T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad D = T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

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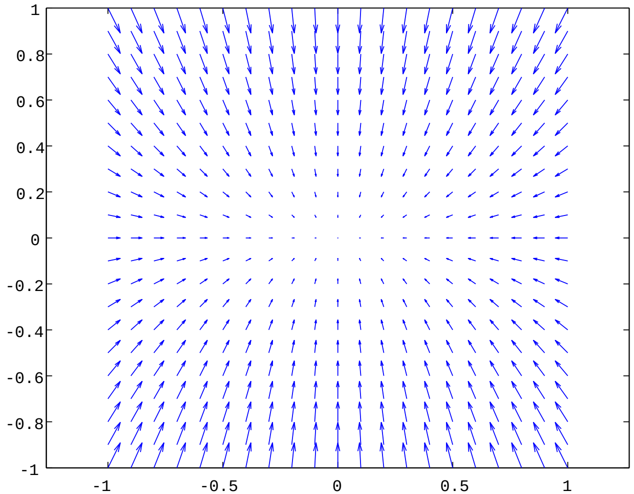
Diagonalization

- Let $z = T^{-1}x$, then we have the diagonalized system

$$\dot{z} = T^{-1}\dot{x} = T^{-1}Ax = T^{-1}ATz = Dz$$

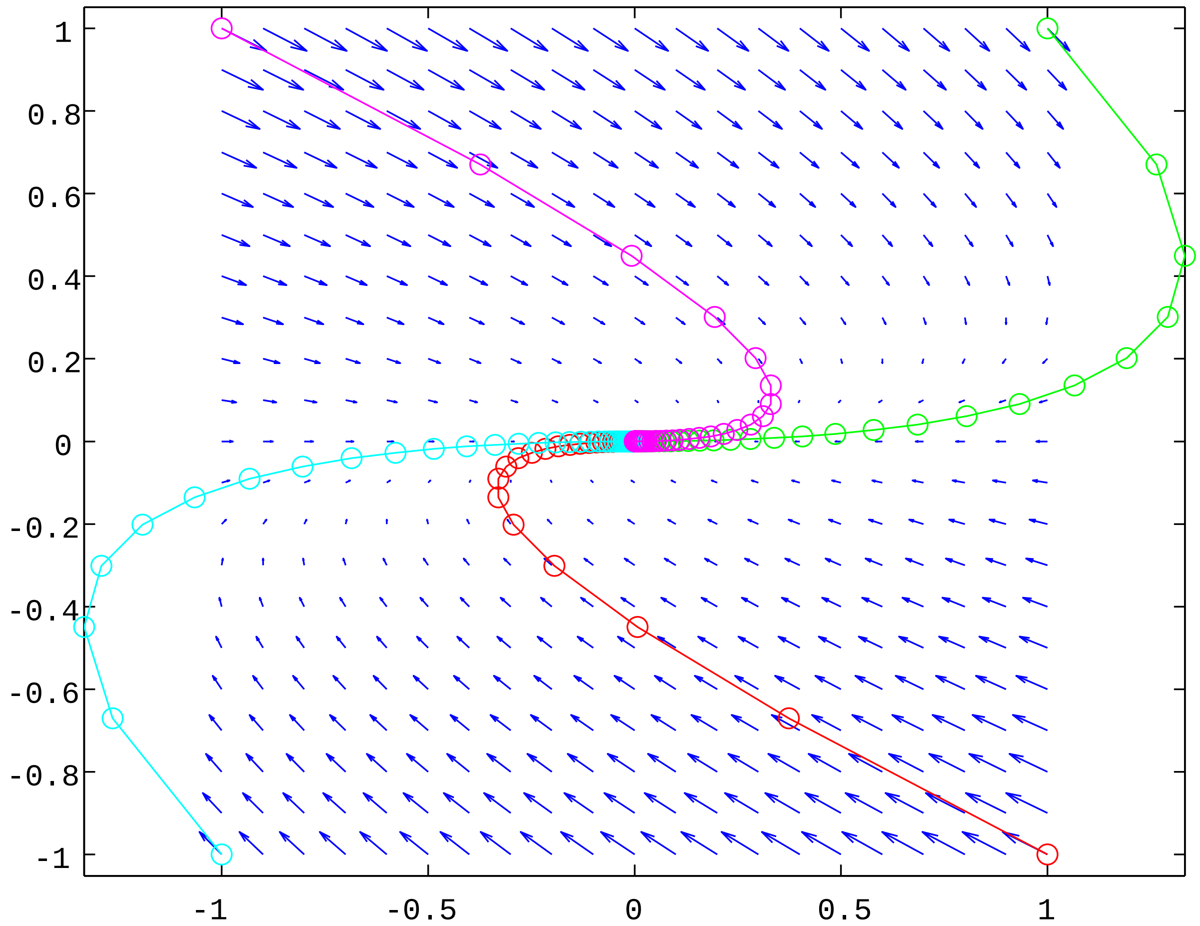
i.e.,

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1, & z_1(t) = z_1(0)e^{\lambda_1 t} \\ \dot{z}_2 &= \lambda_2 z_2, & z_2(t) = z_2(0)e^{\lambda_2 t} \end{aligned}$$



$\lambda_1 = -1,\ \lambda_2 = -2$

- The trajectory is $x(t) = v_1 z_1(t) + v_2 z_2(t)$



Example I

- Consider the system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

- Eigenvalues, Eigenvectors

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & -1 \\ 0 & 1/3 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$$

$$x(t) = Tz(t) = Te^{Dt}T^{-1}x(0) = \begin{bmatrix} e^{-t} & 3(e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} x(0)$$

```
>> [T, D]=eig(A,'nobalance')
>> inv(T)
```

Example I: Commands to obtain vector field

```
% QUIVER(X,Y,U,V) plots velocity vectors as arrows with
% components (u,v) at the points (x,y).

[x, y]= meshgrid(-1:0.1:1, -1:0.1:1);
px=-x+3*y;
py=-2*y;
quiver(x,y,px,py);

% And trajectory starting from (x10, x20)

t=0:0.1:10; x10=1; x20=1;
plot(x10*exp(-t)+x20*3*(exp(-t)-exp(-2*t)),
     x20*exp(-2*t),'go-');
```

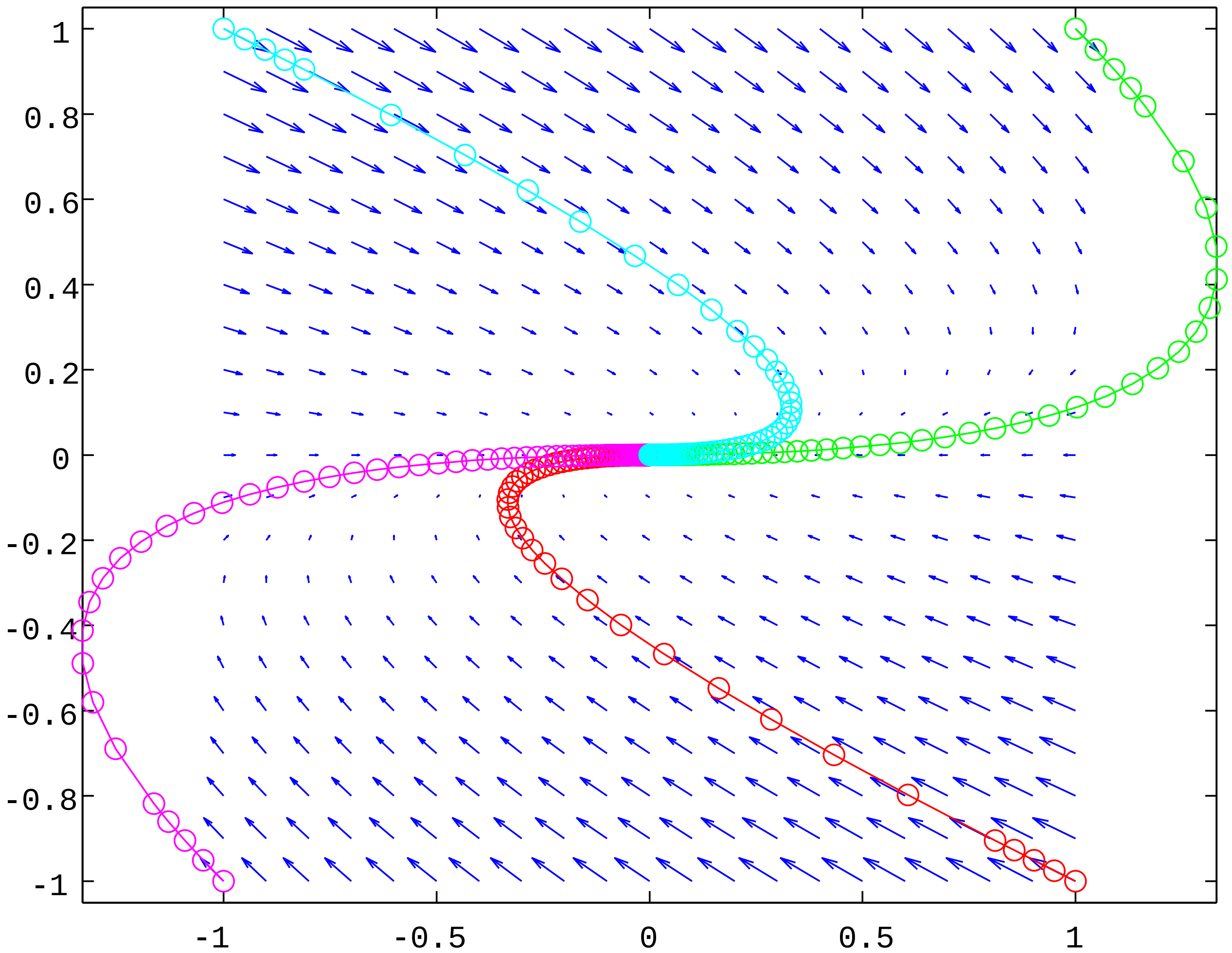
Example I: Commands to obtain vector field

```
[x, y]= meshgrid(-1:0.1:1, -1:0.1:1);
px=-x+3*y;
py=-2*y;
quiver(x,y,px,py);
hold on;

% [T,Y] = solver(odefun,tspan,y0) integrates the system of
% differential equations for tspan=[t0 tf] with initial
% conditions y0
[t,xx]=ode45(@func_d,[0 10],[1;1]);
plot(xx(:,1),xx(:,2),'go-');

% function definition
function dx = func_d(t, x)
dx = [-x(1)+3*x(2); -2*x(2)];
end
```

Example I



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Example II

- Consider the system

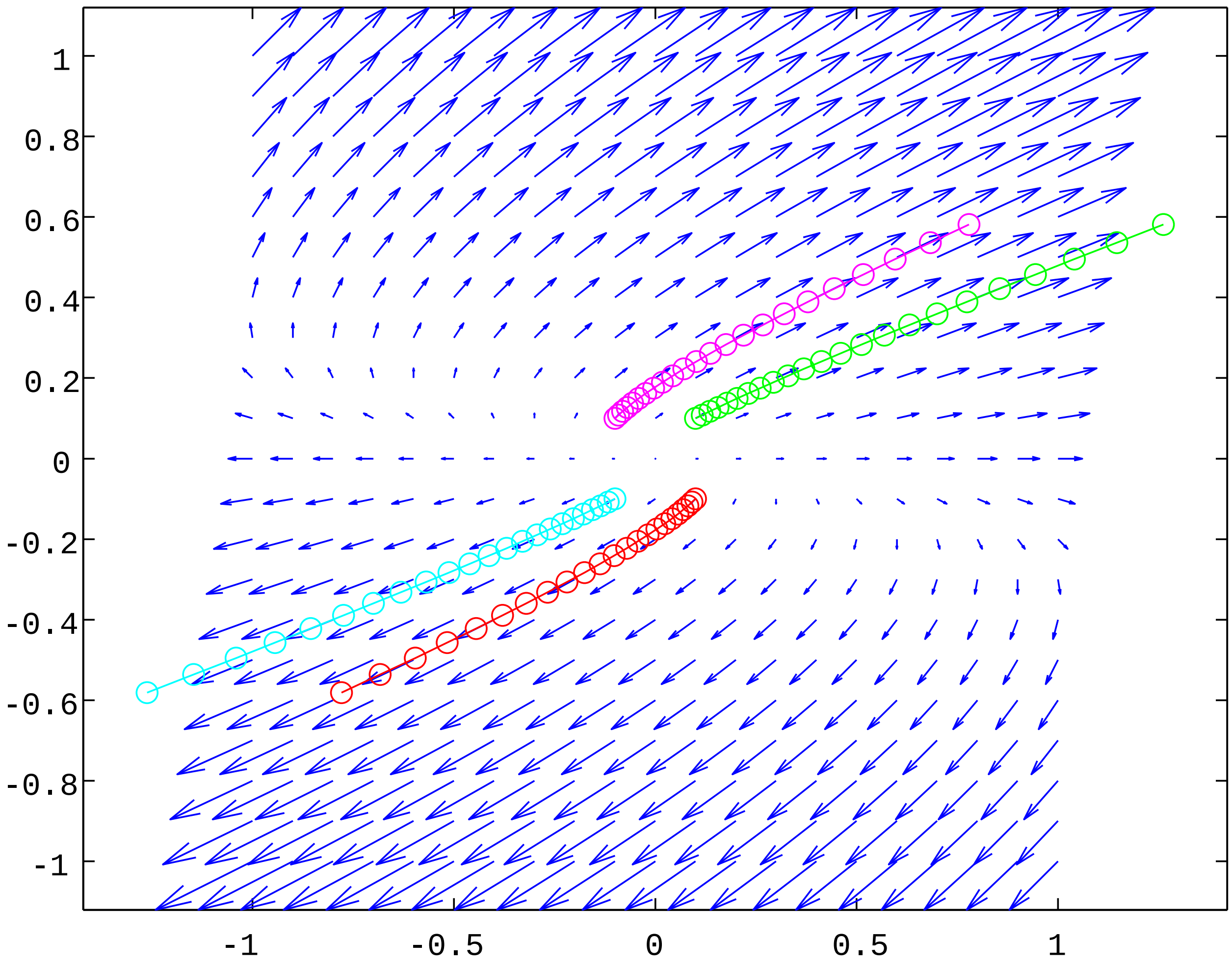
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

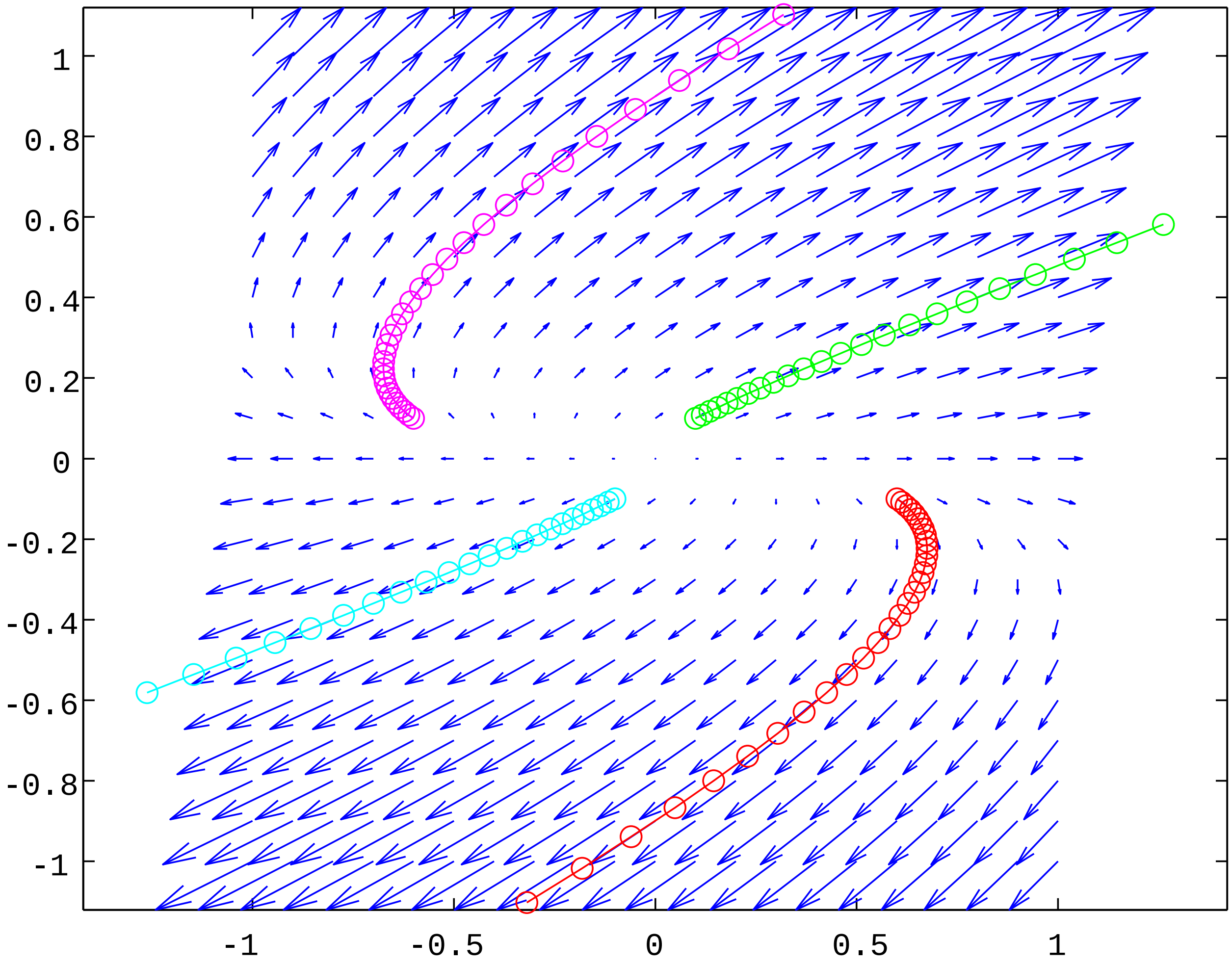
- Eigenvalues, Eigenvectors

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1/3 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 3 \end{bmatrix}$$

$$x(t) = Tz(t) = Te^{Dt}T^{-1}x(0) = \begin{bmatrix} e^t & -3(e^t - e^{2t}) \\ 0 & e^{2t} \end{bmatrix} x(0)$$

Example II




1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1

Example III

- Consider the system

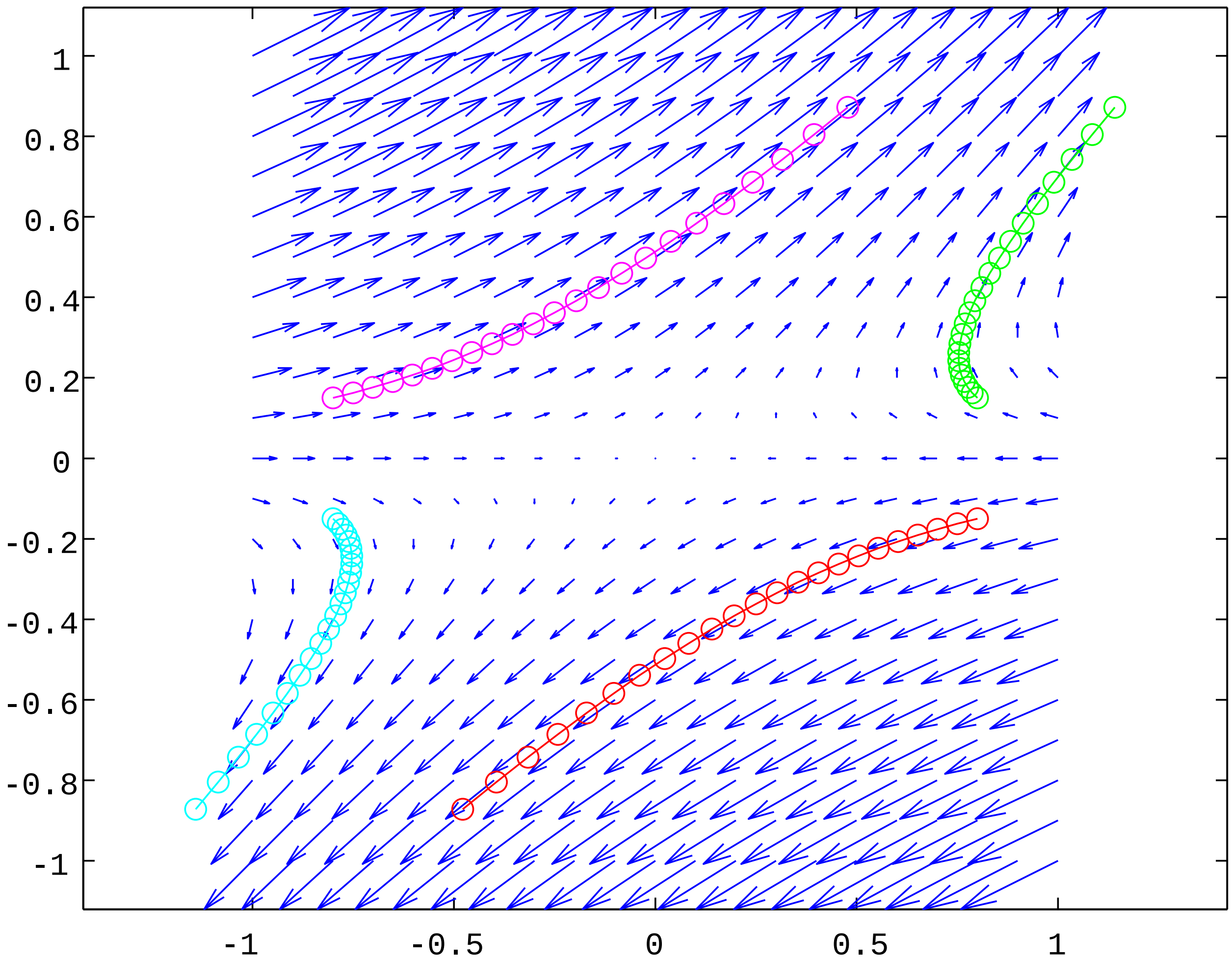
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Eigenvalues, Eigenvectors

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

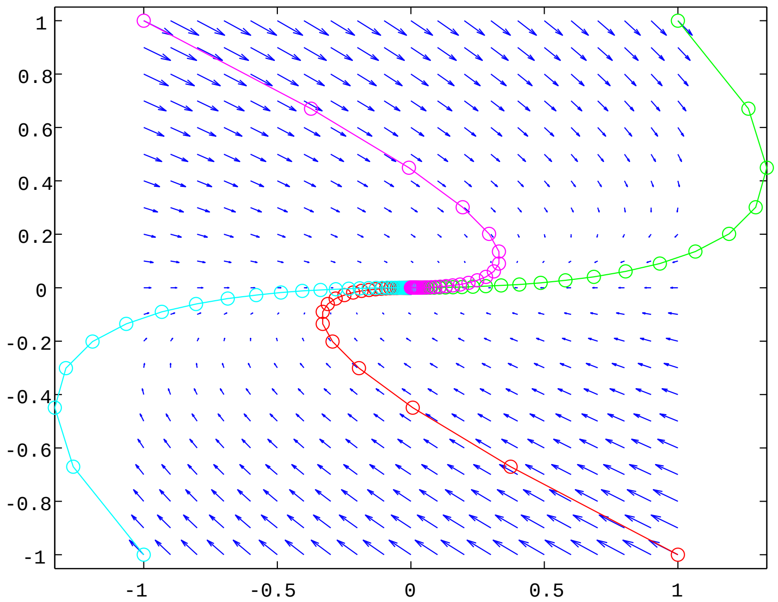
$$x(t) = Tz(t) = Te^{Dt}T^{-1}x(0) = \begin{bmatrix} e^{-t} & -e^{-t} + e^{2t} \\ 0 & e^{2t} \end{bmatrix} x(0)$$

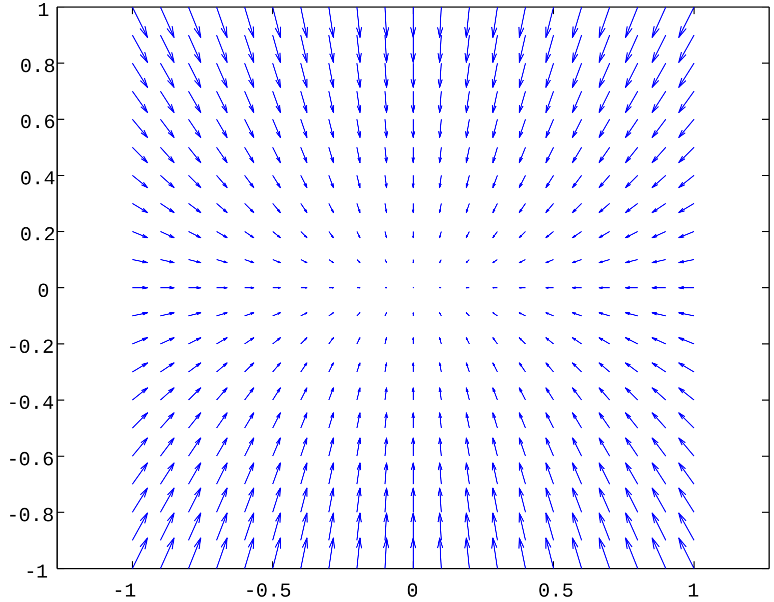
Example III



LTI Systems: Diagonalizable I

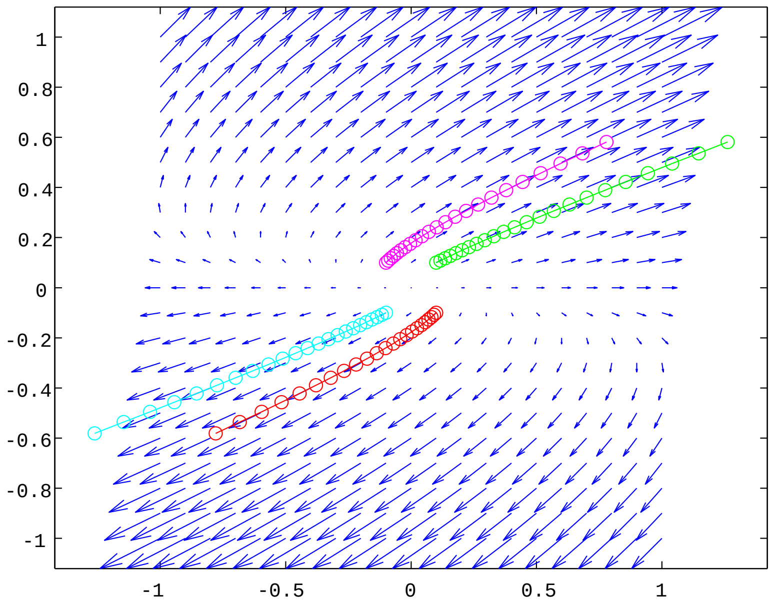
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

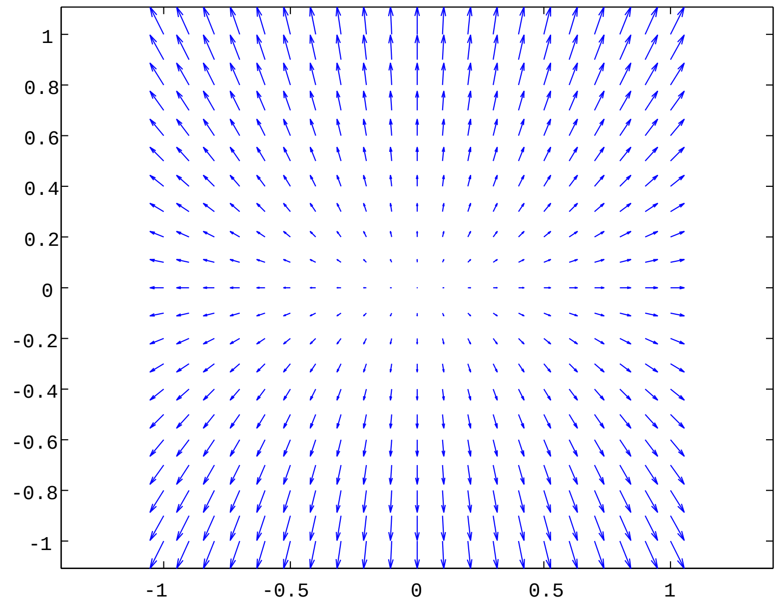




LTI Systems: Diagonalizable II

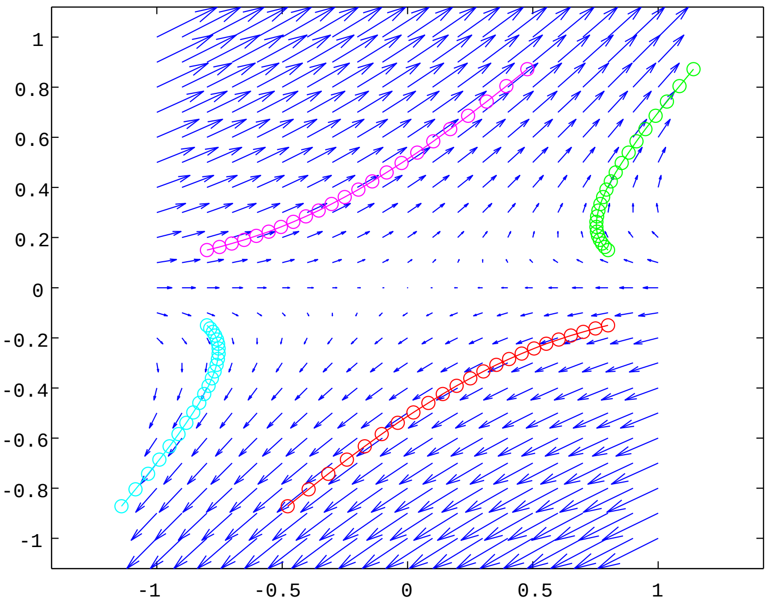
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

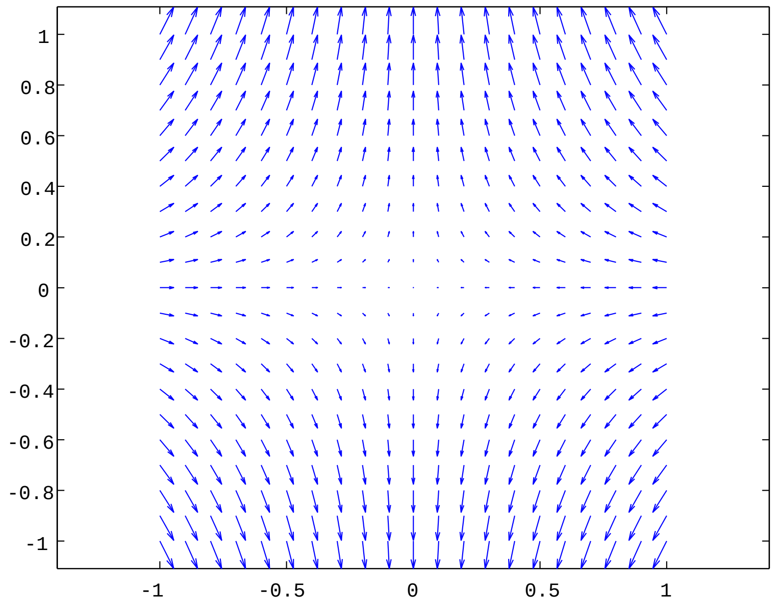




LTI Systems: Diagonalizable III

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$





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LTI Systems: Nondiagonalizable

- For duplicated eigenvalues, we have the Jordan form

$$T^{-1}AT = J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix}$$

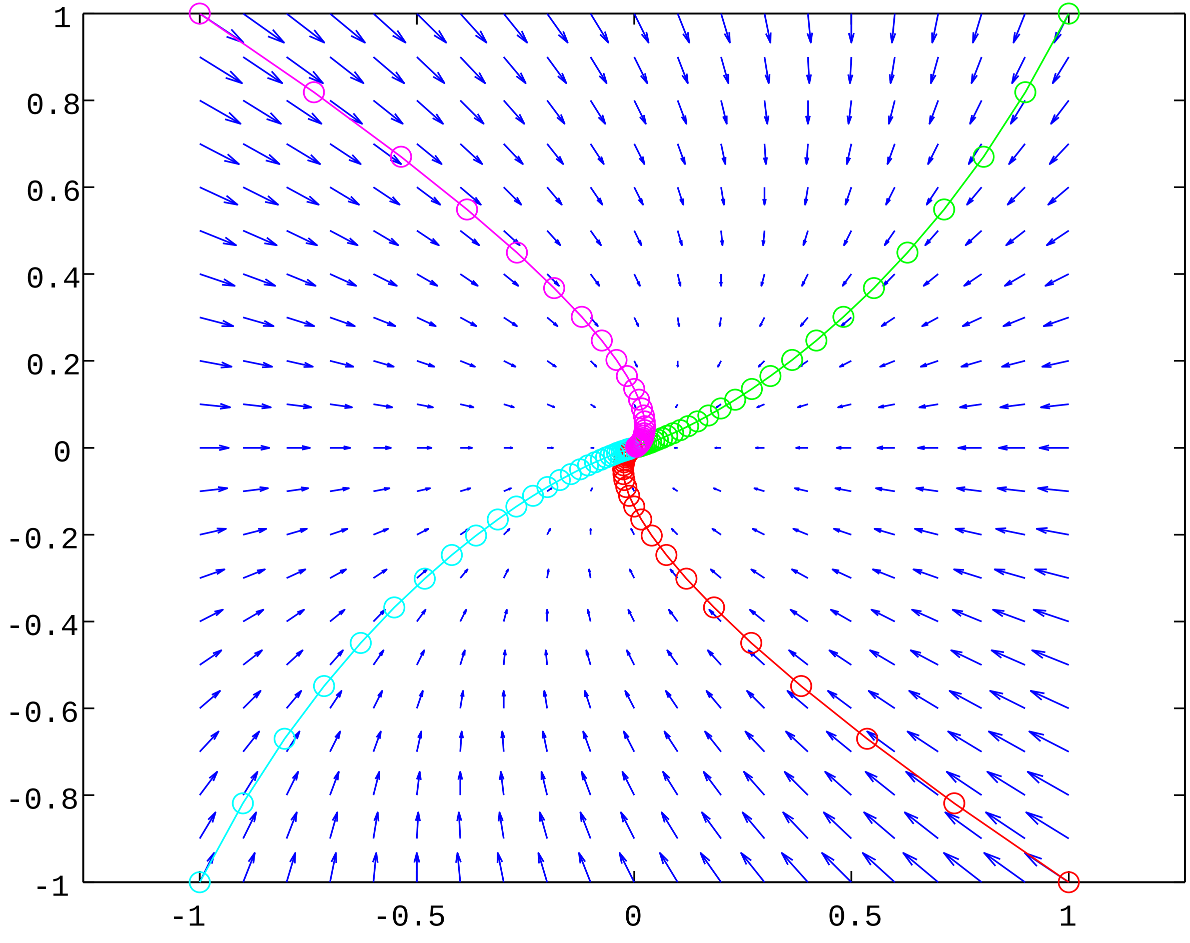
- Consider the system (Eigenvalues: $\lambda_1 = \lambda_2 = -2$)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{x}_1(0) = -2x_1(0) + x_2(0)$$

$$\begin{aligned} x_1(t) &= x_1(0)e^{-2t} + x_2(0)te^{-2t} \quad \text{(check!)} \\ x_2(t) &= x_2(0)e^{-2t} \end{aligned}$$

LTI Systems: Nondiagonalizable



LTI Systems: Complex Conjugate

- If the eigenvalues of the matrix A are complex conjugate, $\lambda_{1,2} = \alpha \pm j\beta$, we can have

$$T^{-1}AT = Q = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

for real α and β.

- Thus the transformed system has the form

$$\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \beta z_1 + \alpha z_2 \end{aligned}$$

LTI Systems: Complex Conjugate

- Polar coordinates:

$$\begin{aligned} \rho &= \sqrt{z_1^2 + z_2^2} & \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \theta &= \tan^{-1}\left(\frac{z_2}{z_1}\right) & \dot{z}_2 &= \beta z_1 + \alpha z_2 \end{aligned}$$

- System in polar coordinate:

$$\begin{aligned} \dot{\rho} &= \frac{1}{2}\frac{1}{\sqrt{z_1^2 + z_2^2}}(2z_1\dot{z}_1 + 2z_2\dot{z}_2) = \alpha\rho \\ \frac{d\tan\theta}{dt} &= (1 + \tan^2\theta)\dot{\theta} = \frac{\dot{z}_2 z_1 - \dot{z}_1 z_2}{z_1^2} = \frac{\beta(z_1^2 + z_2^2)}{z_1^2} \\ \dot{\theta} &= \frac{\beta(z_1^2 + z_2^2)}{z_1^2(1 + \frac{z_2^2}{z_1^2})} = \beta \end{aligned}$$

LTI Systems: Complex Conjugate

- System in polar coordinate:

$$\begin{aligned} \dot{\rho} &= \alpha\rho \\ \dot{\theta} &= \beta \end{aligned}$$

- ρ increases exponentially, decreases exponentially or stays constant depending the real part α of the eigenvalues $\lambda_{1,2}$.
- The phase angle increases linearly with a velocity β, imaginaly part of the eigenvalues $\lambda_{1,2}$.

- When $\alpha > 0$, the trajectories diverge: **unstable focus**.
- When $\alpha < 0$, the trajectories converge toward the origin: **stable focus**.
- When $\alpha = 0$, the trajectories make **closed ellipses**.

Complex Conjugate: Example I

- Consider the following system:

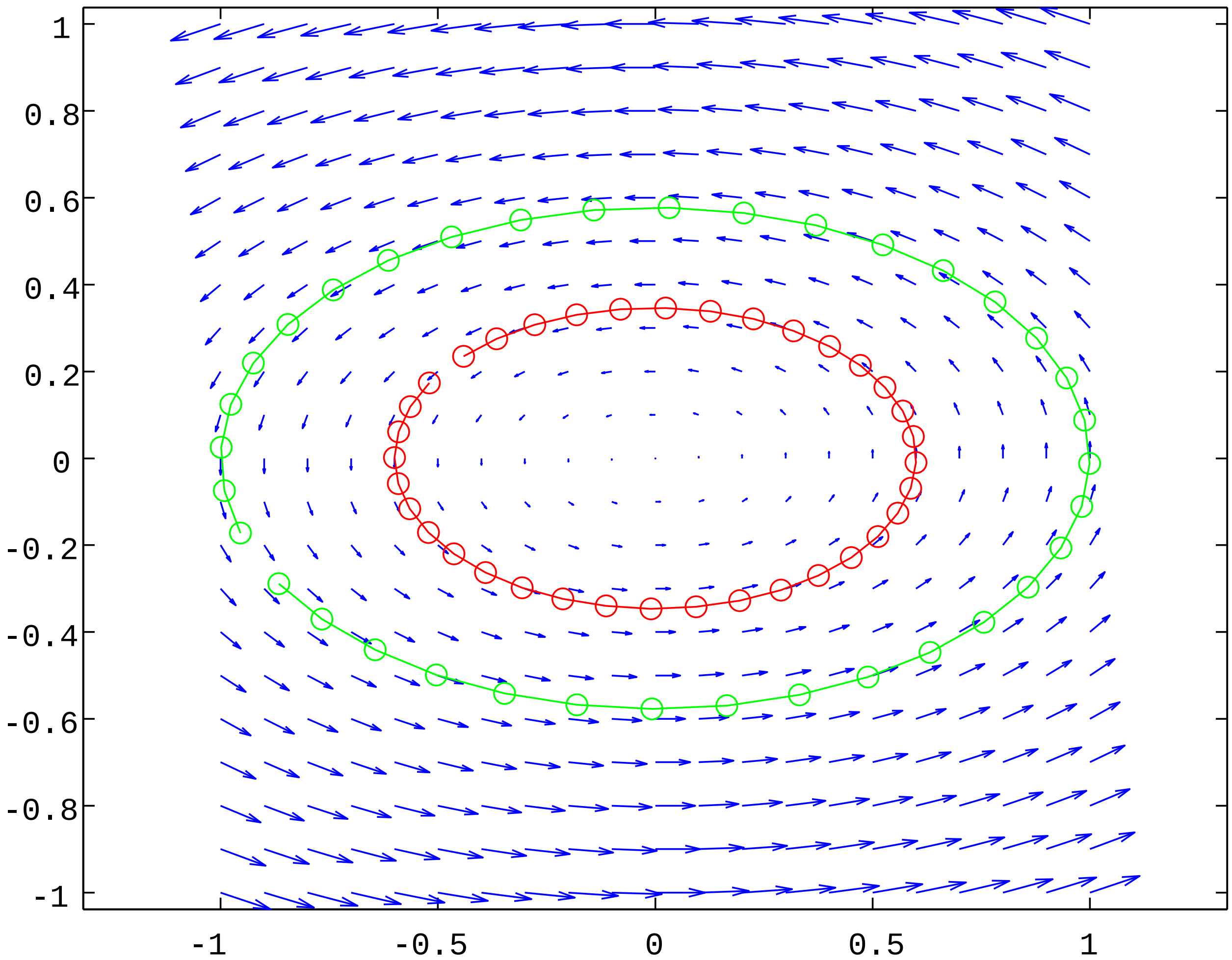
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \lambda_{1,2} = 0 \pm j\sqrt{3}$$

$$x(t) = Tz(t), \quad T = \begin{bmatrix} 0 & -\sqrt{3} \\ 1 & 0 \end{bmatrix}, \quad T^{-1}AT = \begin{bmatrix} 0 & -\sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}$$

$$\begin{aligned} \dot{\rho} &= 0, \quad \rho(t) = \rho(0) = \sqrt{x_1(0)^2 + x_2(0)^2}, \\ \dot{\theta} &= \sqrt{3}, \quad \theta(t) = \sqrt{3}t + \theta(0) = \sqrt{3}t + \tan^{-1}\frac{x_2(0)}{x_1(0)} \end{aligned}$$

$$\begin{aligned} x_1(t) &= -\sqrt{3}z_2(t) = -\sqrt{3}\rho(0)\sin(\theta(t)) \\ x_2(t) &= z_1(t) = \rho(0)\cos(\theta(t)) \end{aligned}$$

Complex Conjugate: Example I



Complex Conjugate: Example II

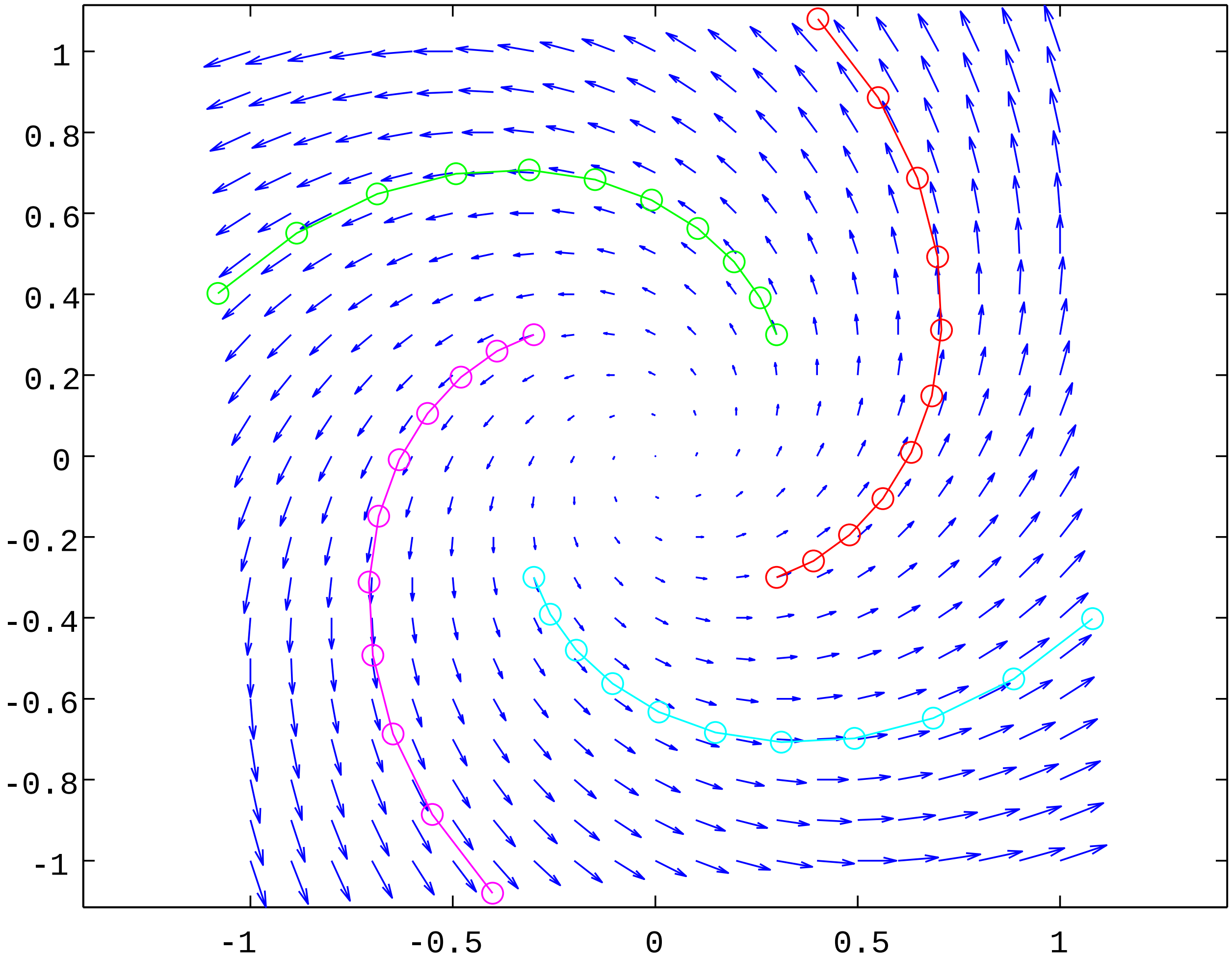
- Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0.5 & -1 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \lambda_{1,2} = 0.5 \pm j$$

$$\begin{aligned} \dot{\rho} &= 0.5\rho, \quad \rho(t) = \rho(0)e^{0.5t}, \quad \rho(0) = \sqrt{x_1(0)^2 + x_2(0)^2}, \\ \dot{\theta} &= -1, \quad \theta(t) = t + \theta(0) = t + \tan^{-1}\frac{x_2(0)}{x_1(0)} \end{aligned}$$

$$\begin{aligned} x_1(t) &= \rho(0)e^{0.5t}\cos(\theta(t)) \\ x_2(t) &= \rho(0)e^{0.5t}\sin(\theta(t)) \end{aligned}$$

Complex Conjugate: Example II



Summary of LTI Systems

Eigenvalues	Equilibrium point
$\lambda_{1,2}$ real and negative	stable node
$\lambda_{1,2}$ real and positive	unstable node
$\lambda_{1,2}$ real, opposite signs	saddle
$\lambda_{1,2}$ complex with negative real part	stable focus
$\lambda_{1,2}$ complex with positive real part	unstable focus
$\lambda_{1,2}$ imaginary	center

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- **System**: A set of first-order ordinary differential equations

$$
\begin{aligned}
\dot{x} &= f(x, t, u) \\
y &= h(x, t, u)
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_n, t, u_1, \ldots, u_p) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n, t, u_1, \ldots, u_p) \\
y_1 &= h_1(x_1, \ldots, x_n, t, u_1, \ldots, u_p) \\
&\vdots \\
y_m &= h_m(x_1, \ldots, x_n, t, u_1, \ldots, u_p)
\end{aligned}
$$

Example I

- Consider the following system where r is a parameter.

$$\dot{x} = -r + x^2$$

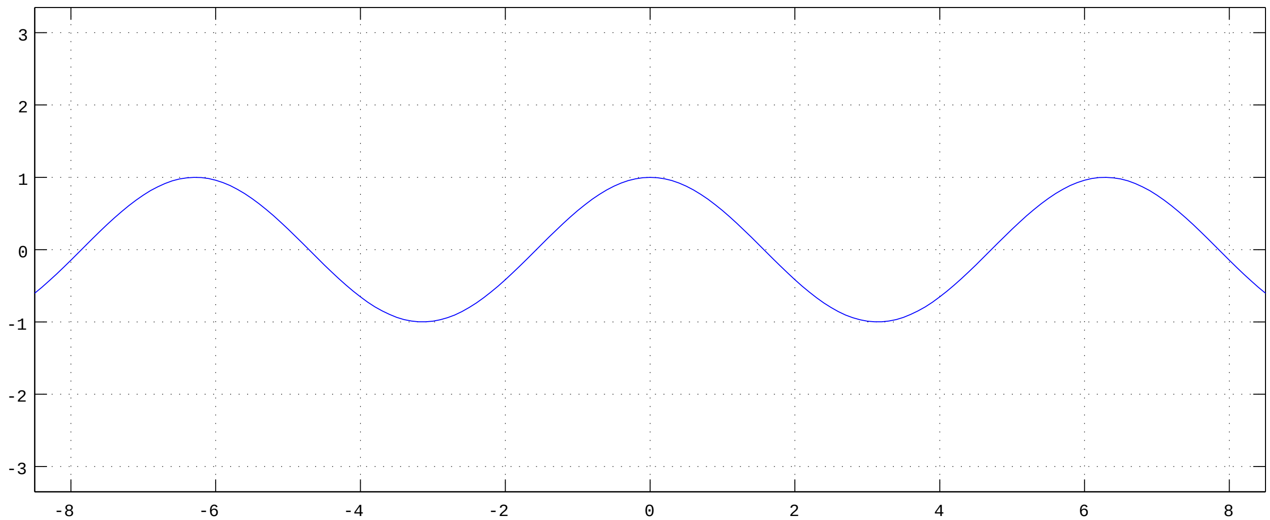
1. If $r < 0$, the system has two equilibrium points $x = \pm\sqrt{r}$.
2. If $r = 0$, both of the equilibrium points collapse, the equilibrium point is $x = 0$.
3. If $r > 0$, then the system has no equilibrium points.

Example II

- Consider the system

$$\dot{x} = \cos x$$

1. The points where $\dot{x} = 0$ are equilibrium points.
2. Whenever $\dot{x} > 0$, the trajectories move to the right, and vice versa.



- Consider the system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}$$

- The solution of the differential equation with a initial condition $x_0 = [x_{10}, x_{20}]$ is called a trajectory from x_0.
- The trajectory presented in x_1–x_2 plane is called **phase-plane**.
- $f(x)$ in

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = f(x)$$

is called a **vector field**.

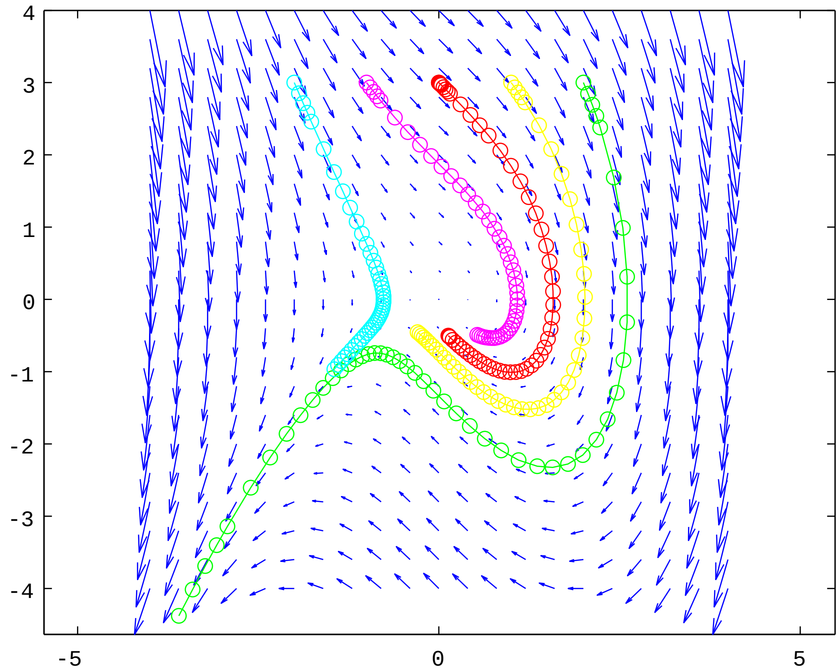
Vector Field Diagram

- To each point x^* in the plane we can assign a vector with amplitude and direction of $f(x^*)$.
- For easy visualization we can represent $f(x)$ as a vector based at x, i.e., we assign to x the directed line segment from x to $x + f(x)$.
- Repeating this operation at every point in the plane, we obtain a **vector field diagram**.

Vector Field Diagram

- Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^2 - x_2$$



- Find a nonlinear system in your research field and derive a state equation (a set of nonlinear **first-order** ordinary differential equations):

$$\dot{x} = f(x, t, u), \quad y = h(x)$$